THE MATHEMATICS TEACHER

EDITED BY
W. H. METZLER

ASSOCIATED WITH

EUGENE R. SMITH HARRY D. GAYLORD
GEO. GAILEY CHAMBERS WILLIAM E. BRECKENRIDGE

VOLUME XI JUNE, 1919 NUMBER 4

TESTS OF MATHEMATICAL ABILITY—THEIR SCOPE AND SIGNIFICANCE.

BY AGNES L. ROGERS.

Measurement in the field of algebra and geometry is of such recent origin, it may at first glance seem somewhat premature to survey the work so far accomplished. It was only in 1914 that the first tests of ability in high school mathematics were published. The production, however, although so recent, has been so suggestive for future work that the time is undoubtedly propitious for taking stock of the tests at our command and for estimating their significance.

As in the history of educational measurement in general, one of the primary aims in the construction of tests of mathematical ability has been the establishment of standards. The admirable results already achieved by standardization in the elementary school subjects have naturally had considerable influence upon those, who desire to see measurement applied to high school work also. Consequently, we find that the first tests of mathematical ability were devised with a view to standardizing the simplest branch of the subject, namely ninth-grade algebra and that the technique employed has followed closely the methods adopted in the measurement of arithmetical abilities.

Thus in both cases use is made of the same two distinctive types of tests, based on methods of measuring that were first applied in psychology, but refined and improved in accordance

with the development of the mathematics of the measurement of variables and also in accordance with the special purposes in view. One type gauges in terms of amount done primarily the ability to perform a uniform task. It emphasizes speed. Thus we have in first-year algebra, corresponding to the Courtis Standard Tests in arithmetic, tests of the same fundamental description by Monroe, Childs, Stromquist, and Rugg and Clark. Of these the two last-named writers have made by far the most elaborate inquiry into the achievement of pupils in ninth-grade algebra. They have had the assistance of several other workers in investigating in detail the many related questions that bear upon the problem; but it is the tests, which they have devised that have yielded the crucial evidence in determining what reconstruction is necessary. While they have carried out a much more extensive inquiry than these other investigators, their tests are in general the same in principles of construction, such differences as exist being due to the much more careful analysis of the mental functions involved in first-year algebra, the more comprehensive and representative abilities measured and the more economical devices applied.

The other type of test is the so-called difficulty, or development, or power test, which measures primarily how hard a task the pupil is able to perform. It consists of a number of problems, which present, as it were, a series of hurdles each exceeding the one before by a definitely determined increment of difficulty. Thus we have in algebra, corresponding to the Woody scales in arithmetic, Hotz's first-year algebra scales and Coleman's scale for testing ability in algebra. The makers of scales, while aiming at constructing a measuring rod having an absolute zero point, a definite unit of measurement and equivalent increments of difficulty, such as we have in our measuring instruments for physical phenomena like the yard stick or the thermometer, have in actual fact attained only approximations to such scales; yet while it is true that they are not perfect from the standpoint of the ideal measuring rod, these first-year algebra scales are nevertheless exceedingly valuable, if only because they furnish us with objective and relatively accurate measures. Thus the difficulty of each step has been calculated

and fairly reliable standards of class performance have been determined.

The grading of the problems in the scales by Hotz and Coleman is based upon the proportion of pupils successfully solving each. Obviously where 80 per cent. of a group of pupils can successfully solve one problem and only 60 per cent. of the same group can successfully solve another, the latter must be a harder problem than the former. We can determine precisely the relative difficulty from our knowledge of the nature of the distribution of mental abilities in general in any random sample of human beings. This, however, is not the sole method that has been employed to grade problems in difficulty. In the first tests in algebra and geometry to appear, which were those published by Professor Thorndike in the MATHEMATICS TEACHER in 1914, the scaling of the problems was based not on the proportion of pupils solving each, but on the judgment of a group of two hundred teachers of mathematics as to their relative difficulty. In this instance Professor Thorndike was interested primarily in demonstrating to those who acted as judges the statistical procedure underlying the judgment scale in distinction from the performance scale. The former depends upon the principle described first in 1898 by Fullerton and Cattell, namely that equally often noted differences are equal.

To take a concrete case, where 50 per cent. of a group of experts judge that writing specimen A has more merit than writing specimen B and the remaining 50 per cent. hold the opposite opinion, then the two specimens are inferred to be of equal merit. We find that judgments are so distributed in cases where objective measurement is also possible, as in judging the relative length of lines. Where 75 per cent. of a group of experts state that writing specimen A has more merit than writing specimen B, and the remaining 25 per cent. support the contrary view, then writing specimen A is held to surpass writing specimen B by one unit of merit. In the scale so derived, the grading of the specimens depends upon the proportion of judges estimating one sample to be superior to another. In the instance under discussion a representative series of algebraic problems was collected. These were then compared each with

every other by the two hundred teachers and so a scale representing the trend of their judgments constructed.

This method is properly applied in the case of English composition, drawing or writing, where the differences in merit *must* ultimately rest upon the judgment of trained jugdes; but the more objective method of measurement is preferable, wherever it is available. It is obvious that in algebra the performance scale should be used. We still can attribute some value to the judgment scale, however, even while granting its relative inferiority in the case of algebra, since it does yield an instrument for rough measurement at least.

The two main types of measuring rods—the standard test and the scale,—have their distinctive uses in accordance with their mode of construction. Thus the scale essentially consists of a series of problems, which although originally chosen with a view to securing as varied and inclusive a group as possible, finally retains only those which happen to be correctly solved by a proportionately increasing number of pupils. Thus the ultimate basis of selection is statistical. A test constructed in such a manner can scarcely avoid losing its original comprehensive and representative character. However varied the cases constituting the series in its first form may have been, in its final form it can hardly fail, as a result of this method of elimination, to lack many problems, with regard to which we need to know specifically, whether they can be solved by any children in a particular grade and precisely by what percentage or proportion. Measurement is necessarily inadequate, when in constructing instruments, the mental processes involved in learning are regarded as of secondary importance.

In this respect scales present a striking contrast to the best types of standard tests. In their case analysis is first made of the mental processes included in the field of study in question and examples of each type are incorporated. This implies a long and difficult task and it is a splendid achievement that we already have such an analysis of ninth-grade algebra. We had previously been unable to define in detail the ground covered as the formation of such and such specific mental bonds between such and such situations and such and such responses. At last now as a result of analytic study the mental associations and dis-

sociations constituting elementary algebra have been determined. We have hitherto thought vaguely in terms of complex mental capacities such as ability to add, ability to multiply, ability to factor, or ability to solve an equation and the like, rather than of the more elementary bonds, of which those composite abilities are composed. The breaking up of those complexes into specific stimulus-response connections is of immense value; since successful teaching really consists in establishing these mental associations in pupils to the required degree of strength, in the best order and in minimal time. In other words, on this analysis all economical learning and teaching depend.

It is to the psychologists we owe the realization of the importance of analysis. Psychologists of every school—analytical psychology, functional psychology, behaviorism or what not—have this in common; they make an intensive analysis of the material they study—namely mental functioning. In ordinary life it is only too apparent that as a rule we fail to give any analytical consideration to the tasks with which we are confronted. These appear to us as units, literally en bloc. Such an attitude, however, leads to ineffective work and wasteful expenditure of energy. We can short-cut the lengthy and prodigal trial-and-error, hit-or-miss process of learning by a thorough-going analysis of the task, and of the most favorable attitudes for work. By such methods Gilbreth was enabled to treble the work of bricklayers and many other students of scientific management have discovered the sources of loss, waste and fatigue by similar means. Just as in industry to-day the efficiency engineer has by painstaking study reduced motor processes to their simplest elements, has determined the essential constituents of the skill in question, thereby eliminating useless and superfluous movements, has determined the best order of their formation and the amount of practice needed for perfecting each item in the total processes; so in the establishment of algebraic skill, analysis is an indispensable precondition and if the tests had accomplished nothing further, this would have been in itself a noteworthy contribution. It is a great step in advance to know the requirements of first-year algebra in detail from the standpoint of learning, from the standpoint of the pupil and the changes we are attempting to effect in him. We

shall make advance in improving the teaching of algebra precisely in proportion as we succeed in making such analyses of its complex character.

The tests devised by Rugg and Clark stand out preeminently in this respect. They represent an analysis of ninth-grade algebra in to the unitary abilities, which compose it. They literally test one by one the strength of the bonds that together constitute ninth-grade algebra. A similar analysis of the work for higher grades still lies ahead of us.

Should economy of time demand a more restricted number of problems than we would have, if we included each distinct mental operation in our tests, the device of a spiral arrangement is used, by which several related operations are combined in the same test in cyclic order. This device first used by Judd in the Cleveland Survey arithmetic tests has been incorporated by Rugg and Clark in their algebra tests. To take a concrete example, the ability involved in removal of parentheses can be resolved into six fundamental types of mental associations. Accordingly, in the test to measure this complex capacity these six forms recur at regular intervals. It is worth noting some of the advantages of this device. One is that the score made by one pupil can be compared with that made by another. For example, if boy A attempts 24 problems and has 22 right answers and boy B attempts 12 problems and has 11 right answers, then we are justified in asserting that boy A has accomplished twice as much as boy B. While every mental bond or manipulation is not accounted for in the test so designed, at least the essentially new elements in learning to remove parentheses are included. Another advantage a spiral order offers is that it enables us to distinguish errors due to accident from errors due to ignorance or incapacity. Errors of the latter sort will recur at regular intervals and so be readily recognized. By this arrangement, too, we have a means of ascertaining not only errors common to the class as a whole; but also of locating individual errors as well and this entails the important consequence that thereby we are furnished with information indispensable for proper teaching emphasis both in the case of the class as a group and of the pupils individually.

It must be self-evident that this is one of the most vital uses

to which tests can be put. When we have determined the class accomplishment and have compared it with those of other classes in other cities or with its own previous performance and so measured progress, interesting though that information may be, it has still to be interpreted. Should we find the results so obtained are unsatisfactory, this should merely instigate us to that further inquiry into the sources or causes of the condition revealed. The discovery of the latter is the greater discovery, which alone can suggest in what direction action is necessary. In addition to the help the tests can give in securing accurate measures of individual skill and of class performance and thus in improving current methods of classifying pupils, they can by diagnosing individual weaknesses and class weaknesses suggest radical alterations in methods of teaching.

Experimentation in arithmetic has demonstrated irrefutably that the recognition of individual errors and their special treatment is the direction along which we may look for the realization of the greatest economy of time. The diagnosis of individual weaknesses is the *first* step in the process of differentiating teaching to meet individual needs and as such forms the keystone in any time-saving program. The increase in efficiency resulting from a comprehensive scheme of adaptation of teaching to individual requirements cannot be fully foreseen, but it is practically certain that it will revolutionize teaching in algebra, as it has already revolutionized teaching in other fields.

What is still more worthy of note, standard tests can modify greatly methods of *learning*, for they are invaluable as furnishing motives for self-initiated activity, especially in connection with the more formal aspects of algebraic study. They act as incentives to zeal on the part of *pupils*, as well as of *teachers*. We all need such standards. We profit greatly by knowing our exact accomplishment, how we compare with our fellows and how our class compares with other classes, for it spurs us on and carries us over the routine work that must be mastered in elementary algebra. Teachers' marks expressed only teachers' opinions of their pupils' work. With the more objective, impersonal and reliable character of the scores derived from tests, pupils are conscious of a more powerful stimulus to endeavor.

They can be brought to realize that a certain standard of proficiency agreed upon by experts must be attained. The effect of hanging a Thorndike handwriting scale in the elementary classroom is the best illustration of the magical way in which standards work, leading the pupil in the case of a routine process to *drill himself,* until the required proficiency is achieved.

There is another purpose which the tests can serve. The statistical results already available are modifying markedly our ideas of what ought to be mastered in first-year algebra. Investigations of the mathematics in use in everyday life as also of the mathematical processes taught during the first year, which are essential for more advanced work in the subject, such as those undertaken by Rugg's collaborators, undoubtedly are essential in furnishing a clue to the new material that must be incorporated in the course of study. Nevertheless, we can reconstruct the content of our course wisely and scientifically, only if we know what has actually been accomplished by the material *now* in use. In other words we must first have accurate records of the results of present instruction.

Experimentation with new material and likewise experimentation with new methods of teaching depend for their success upon reliable means of gauging the effects of changed conditions. Standard tests and scales ring the death knell of mere guess-work and they hasten the demise of both rule-of-thumb methods and haphazard change. There will innevitably follow as a result of the more objective and accurate estimation of individual and class achievement, which they permit, the determination not only of minimal but also of maximal essentials, and thereby will be effected great increase of efficiency. For experimentation will reveal the optimum amount of drill, after which further practice would yield diminishing and unprofitable returns and so save time by setting limits of efficiency—maximal points, beyond which the pupil need not go and at the other extreme it will define minimal degrees of attainment, which we have a right to demand from efficient teachers of normal pupils. If this can be done in ninth-grade algebra and it has been in large part accomplished, it can also be done in more advanced work, in which we have every right to expect as fruitful results.

Standardization has also been begun in geometry, although nothing so thoroughgoing or ambitious as Rugg and Clark's algebra tests has been attempted. Four tests of geometrical abilities have been constructed by Minnick. In this investigation the capacities fundamental to the demonstration of a theorem were selected for standardization, the subject-matter embraced being limited to the first two books of geometry. Three of these tests certainly measure abilities of first-rank importance; namely first the ability to draw accurate figures for theorems, secondly the ability to state the hypothesis and the conclusion of a theorem in terms of a given figure, that is to state exactly what is given and what is to be proved in the case of a specified proposition, thirdly the ability to select and organize facts to produce a proof. The fourth test which gauges the ability to recall related facts about figures, when one or more facts are given, appears to lack the definite purposiveness that characterizes true thinking and would seem to be less significant.

For these tests reliable standards have been calculated and they serve for geometry the same valuable ends that standard algebra tests have been shown to serve for algebra. It is by such intensive study of the abilities involved in geometrical work that improvement in the teaching of geometry will take place most rapidly. We are fortunate in being promised a scale for measuring pupils' ability to demonstrate geometrical theorems by the same author in a forthcoming issue of the *School Review*.

There are many other abilities essential to efficiency in elementary geometry besides those to which Minnick has directed his attention and some of these seem to me to be quite important. In particular, one group of capacities, which greatly influence success in elementary geometry is overlooked, namely the various functions included under intuitive grasp of spatial relations. The proof that two triangles are similar depends upon the power to apprehend immediately the spatial relations underlying the application of the principle of superposition. Individuals differ by native endowment as also from training in the dexterity with which they can accomplish this. A means of measuring the degree, to which it has been developed would be

of great value in throwing light upon the causes of geometrical weakness or talent in any student. I have used the Thurstone Spatial Relations test, which in its original form measured the facility with which the pupil could put into practice the principle of symmetry, to estimate the ability to apply both principles, superposition as well as symmetry.

The need of an adequate measuring instrument for space intuition is still greater, when the student encounters for the first time solid geometry. We are familiar with the initial difficulties of handling tridimensional space. Here too great individual differences due to endowment as well as to practice exist. For diagnosis of special individual weakness a standardized test would obviously be of great service and it was for this reason I constructed a measure of the ability involved in intuitive grasp of tridimensional space. When we realize the important rôle spatial intuition plays in advanced mathematics, for example, in analytical geometry, we see that some method of gauging its strength is of significance not only for elementary geometry, but also for its prognostic power as regards future work.

In the *School Review* for October and November Irwin presents a "*Preliminary Attempt to devise a test of the Ability of High School Pupils in the Mental Manipulation of Space Relations.*" Three hundred and thirty-seven pupils in three high school classes were examined in one test and three hundred and fifty-two in another. The material was not strictly geometrical; but the classification of errors is valuable for teachers of geometry.

It may be entirely fortuitous that standardization has not proceeded so far in geometry as in algebra; but no one can fail to recognize the comparative difficulty of isolating specific abilities in geometry, which is a necessary preliminary to designing suitable measuring instruments. Standardization is, indeed, a much harder task in geometry than standardization in algebra in virtue of the obstacles that confront us in resolving geometrical ability into the unitary abilities that compose it. It should be remembered that until recently we have been comparatively ignorant of the mental functions active in geometrical thinking. Whereas algebraic ability is readily differentiated into a number

of easily distinguishable capacities, geometrical ability is more complex and presents less obvious lines of psychological demarcation and moreover the work done in arithmetic yielded helpful suggestions as to the realtive importance and the intrinsic significance of several algebraic functions. In elementary geometry this preliminary spadework had still to be done and much of it still lies ahead. Analysis was essential not merely to break up geometrical abiilty into its elementary constituents, but also to determine the efficiency and status of these unitary abilities in geometrical ability, to discover their interrelations and also their connection or lack of connection with other forms of mental capacity. It was essential to know whether there is an outstanding ability, which is the fundamental capacity in mathematical work and whether mathematical talent is such that it can function with approximately equal facility in relation to all kinds of material or whether it is in its very nature specialized and tied down to definite objects and situations. It was imperative for practical insight, that we should have experimental proof that a high degree of skill in algebra is accompanied by a high degree of skill in geometry or the reverse and that geometrical abilities correlate highly among themselves or otherwise. We grasp the full significance of any test, only when we do know its correlations with other tests. The process of determining these interrelations is a lengthy and laborious one; but ultimately it brings its reward. We cannot estimate the import of the reconstructions proposed in mathematics until such facts are discovered.

A fair amount of suggestive writing has been done along this line of analysis. Metzler and Stockard and Carleton Bell have made good suggestions for tests, although they have not attempted to satisfy the canons of rigorous scientific measurement. Indeed, Metzler's aim was merely to differentiate some of the more fundamental activities involved in geometrical work. Stockard and Carleton Bell, however, did design a test, for which as yet no standards have been published. It includes problems demanding the exercise of very varied mental functions for their solution. To it, therefore, there attach the typical drawbacks of the mixed scale. It cannot, for instance, provide us with a complete diagnosis of the gaps in the child's

learning, the precise points at which mastery has not been reached. It has of course some significance, inasmuch as it can provide a rough instrument for classifying pupils in general geometrical ability; but it is not entirely satisfactory for the reasons stated. Anyone who looks forward to constructing geometrical tests would do well, however, to consider the fruitful suggestions that these writers have made.

A more systematic analysis of both geometrical and algebraic ability was published this year, based on strict methods of measurement. The primary aim of this investigation, which I conducted, was the measurement of mathematical intelligence and the obvious preconditions of success were first the analysis of the meaning of mathematical intelligence and second the construction of instruments by which its amount could be calculated.

Psychologists interested in the measurement of general intelligence had already blazed a trail through the jungle of mental measurements in the work they had accomplished. Binet and his followers,—Terman, Yerkes and others, had largely solved the practical problem of devising a measuring rod of general ability. Pearson, Spearman, Thorndike and their pupils had discovered the tools, by which an analysis of the nature of general ability into simpler mental functions could succeed. In measuring mathematical talent both problems arise, since neither mathematicians nor psychologists are agreed as to what the essential constituents of mathematical intelligence are.

After preliminary analysis, based on introspection of the mental processes involved in high school mathematics, tests were devised to measure as many of these as time would permit. These tests furnished a workable instrument for experimental investigation. For example, they served to determine the relative status of the various capacities measured in general mathematical ability. They were deliberately planned to gauge mathematical intelligence in children who had had five months of algebra and five months of intuitional geometry. My thought was that measurement of innate mathematical capacity was more urgent at this time than at any other stage in the pupil's high school career.

It would seem right in a democracy that no child of normal intelligence should be deprived of the opportunity of becoming

acquainted with a realm of thought, which has meant so much for the advancement of science and of civilization as mathematics. Nor should he be deprived of a training in the use of the tools of quantitative thinking, except where he lacks the ability to profit by it. Both from a cultural and a practical standpoint knowledge of the fundamentals of mathematics should not be withheld, since some degree of mathematical mastery is necessary for advanced work in so many spheres. Nor should the child be considered a fit judge of what will be best for his future development, nor indeed should the parents in most cases. We need to safeguard the rights of the child, so that he is not steered away from the high types of work, of which he is capable.

Since the question of educational guidance is so critical, determining the future career of the pupil, allocating him to occupations of one level or another, we should have as reliable a gauge as the wit of man can devise. We require an independent estimate of mathematical capacity, which being as accurate and objective as we can make it, will serve to confirm or to call in question school marks, teachers' judgments and pupils' opinions of what they can perform. After two years' experience with algebra and geometry we would have the school marks in these subjects and the teachers' judgment of the pupil's ability. These are not necessarily the same, since the former are indicative of industry and other characteristics as well as of mathematical capacity. These together with an independent measure of mathematical ability derived from a group of tests would make it possible for a reliable decision to be arrived at whether further study would be profitable for the pupil. An independent estimate is particularly necessary in mathematics because routine processes play so prominent a rôle in elementary mathematical work. We hope that after reconstruction of ninth-grade algebra less time will be devoted to formal processes and more to thought-provoking tasks; but it is unlikely that ninth-grade algebra will not emphasize the mastery of the former to a considerable extent. Hence we need an instrument that will touch mental functions essential in more advanced mathematics and having little or no place in elementary work. This instrument should test other abilities than those

tested by school examinations. It should supplement these and in this way provide us with a more adequate prognosis of mathematical skill and insight.

Consequently in the investigation undertaken the original abilities demanded by high school mathematics were discovered by pyschological analysis rather than the abilities that are largely the result of training. Of course, ultimately it is impossible to measure any ability apart from all training. We reach this limit only as an ideal. Where intelligence is primarily in question, however, problems requiring adaptation to new situations should predominate. From his measuring scale of intelligence Binet at one time eliminated all tasks presupposing instruction, but later he reintroduced them on the grounds that after all, if a child had had an opportunity of coming in contact with certain experiences, it is an index to his intelligence if they have left no trace or contrariwise.

Similarly in attempting to measure mathematical ability, while tests were for the most part of a novel character, demanding adaptation to numerical, algebraical and spatial material of a new sort; there were also included a few tests with which the pupil would certainly succeed better, if he had had previous training. Thus a test of algebraic computation was used as a rough gauge of the pupil's ability to manipulate abstract symbols and obviously the amount of practice received could greatly affect their scores.

In all seventeen tests were used. Of these five were tests of algebraic functions and a sixth, a reasoning test, measured an ability closely akin to algebraic ability. They were measures of ability in algebraic computation, ability to translate verbal statements of problems into algebraic symbolism, ability to match arithmetical progressions and corresponding formulæ, ability to analyze numerical data, to perceive a general rule implicit in them and to apply the principle so derived, and ability to make inferences in the case of purely symbolic material. There was also a test of skill in arithmetic problems. Five were tests of geometrical capacities, namely ability to make inferences with spatial data, ability to grasp intuitively spatial relations, bi-dimensional and tri-dimensonal and lastly ability to generalize in the field of spatial facts. Four were tests of language ability.

These were the well-known mixed relations or analogies test, the familiar logical opposites test, the Thorndike Reading Scales and the Trabue Language Scales. The inclusion of the language ability tests may at first consideration appear strange; but as our suspicion was and as the experimental results showed, the ability to conceive clearly the meaning of the words of any problem is as important an element in its solution as any connection it has with the abstract symbols of algebra.

Some of the results of the investigation are interesting in connection with current disputes. It has been advocated by several writers, notably by Breslich, that geometry and algebra should be fused as far as possible into a single subject. Our results, however, show that geometrical and algebraic abilities are no more closely related, no more dynamically connected with each other than with such superficially disparate abilities, as are measured by the analogies test, the logical opposites test, the Thorndike Reading Scales and the Trabue Language Scales. It is a striking fact that when the scores of the various geometrical, algebraic, and language tests are combined respectively into three groups, the coefficients of correlation between these composites are practically identical. The crude coefficient for algebra and geometry was .38, for algebra and language ability .42 and for geometry and language ability .41. The corrected coefficeints, which probably give the most true amount of interdependence between the mental functions in question yielded similar results. The coefficient for algebra and geometry was .49, for algebra and language ability .56 and for geometry and language ability .56. Apparently algebra and geometry are in no special degree closely akin.

The results further emphatically indicate that a command of the vernacular is an important precondition of success in mathematical work. The ability to comprehend words, to grasp accurately the significance of a problem is as vital an element in its solution as the manipulation of algebraic symbols. Too little heed is paid to this aspect of mathematical learning. Much more time should be given to clarifying quantitative thinking as expressed in language. These results fully substantiate Rugg and Clark's contention that much more effort should be devoted to translation, not only from English into algebraic symbolism,

but also from algebraic symbolism into English. Only by such exercise will the abstract entities we deal with in algebra become familiar tools in the control of the learner.

Another result worthy of note, although merely confirming common sense, is that no single test can be a sufficient index to mathematical intelligence. Psychologists have usually acted on a different assumption and it is an interesting finding that not one of the seventeen tests used correlated at all closely with the best available measure of mathematical capacity. If there is any one dominant mental function in mathematical ability, it is that which is common to those tests, which probe mathematical intelligence most effectively and this seems to be the ability to react to parts of a situation, in other words to analyze and think selectively. Tests having this characteristic quality in marked degree form the best index to the presence of mathematical talent, although it certainly involves many other functions indispensable for success with the subject.

This investigation also showed that great individual differences in mathematical skill and insight exist. The presence of a very wide range of ability was never more clearly demonstrated and the need for classification in accordance with capacity was never more apparent.

A still more striking result and one that made a great impression upon my mind was the evidence it furnished of the ability of the pupils to attack much harder tasks that we commonly assign to them. We seem prejudiced against the child's ability. The extreme instances of sheer incapacity influence us unduly. Again the need for classification according to inborn power shows itself to be urgent. When the question of reconstructing the content of the elementary courses in mathematics is before us, it will be necessary for many of us to take a less depreciatory view of the capacity of the pupil and to ask ourselves whether we have a correct conception of what he can do, rather than to doubt the advisability of introducing more advanced material earlier to the student. I incline to agree with Thorndike that there is no kind of reasoning a child of eleven cannot do, provided the material lies within the realm of his experience. Indeed, it would be easy to find a child of eleven, who could do more efficient mathematical reasoning than many

adults. Opportunities for the gifted should not be lacking, nor should we take so low a view of the ability of the average child.

By determining the functional interdependence of the traits measured by those tests and their relative importance in mathematical intelligence, it was possible to select a team of six tests, which while possessing maximum diagnostic power was also easy of application and economical of time. The prognoses derived from combining the results of these tests closely correspond with the actual future achievements of the individuals tested. In the case of a group of 61 pupils in Horace Mann School, an hour and a half of time spent on the tests gave a coefficient of correlation of .82 with future mathematical accomplishment.

Thus by the aid of this sextet of tests we can predict not with perfect precision, but with a known degree of accuracy the capacity of the pupil to undertake more advanced work. By its use we are able to classify pupils on a more reliable basis, so improving current conditions, by eliminating from the mathematics classes those unfit for further training and selecting those of superior ability for special treatment.

Much more scientific work remains to be done along the line of this investigation. We need tests to measure innate algebraic intelligence, before the child has had any algebraic training whatever. We should have a test or a group of tests of such a character applied the first day in the ninth-grade algebra course, so that we could from the outset group pupils more appropriately, in consonance with their powers and their promise as students of the subject. The same holds good of geometry. For it, too, we need a team of tests or a single test of high diagnostic power, which presuppose no previous knowledge with the subject, and which could provide us with an adequate measure of innate ability in that field.

Prognostic tests to improve classification in our larger high schools, standardized tests for more advanced algebra and geometry—these are the tasks that await our attention. The work will be difficult; but the results will be proportionately precious. Surely in no other subject are rule-of-thumb methods more ridiculous than in the teaching of mathematics. Surely in no circumstances have we a greater right to expect helpful

assistance in measurement than from a group of people trained in quantitative thinking, with much of the equipment that is necessary to carry on the work. Mathematics teachers should lead in this movement to replace subjective and unreliable school marks by objective, unbiased and accurate methods of measuring. We would thereby not only be aiding in the improvement of the teaching of our own subject, but we would also be making of education a science and playing a part in one of the great movements of modern times.

We can scarcely grasp the full significance of the introduction of scientific measurement in education. We can but dimly discern its probable results. When clairvoyance and divination were for the first time in history replaced by science, by the inductive experimental method, how few were conscious of the momentousness of the event that had taken place! How few realized that a turning point in the history of mankind had been reached. In like fashion we apprehend but faintly that a new era has begun in the records of the human race. The birth of physical science was a world-shaping event, inasmuch as it gave control over the material universe. The application of measurement and experimentation to mental products is as remarkable an occurrence, for it literally means control of the human machine. The last factor in efficiency to be controlled, the human factor will in all likelihood yield the richest results of all. It seems only yesterday that we ignored the fact that a few unnecessary movements in an activity like sewing adds much to the bodily energy consumed in a day, so we now are for the most part unaware of the wastefulness of our methods of habit-formation in mathematical work. As soon as effort is devoted to changing this state of affairs, we shall be magnificently rewarded in the saving in human strength that will result. At this time in the world's history, when the minds of men were never more malleable, when far-reaching adjustments in our social and in our educational institutions are in progress, we should be eager to play a part in a revolution still more fundamental, more invasive and pervasive than any other—the downthrow of opinion and the rise of science in the schoolroom. When we recall how long it took before we had precise instruments of measurement in physical science and how much is

expended yearly to maintain bureaus of weights and measures, we shall not grudge the cost of obtaining educational measuring rods, although the magnitude of the task confronting us may temporarily give us pause. It is well to remember, however, that in no subject are circumstances more favorable, in none should progress be so rapid as in mathematics. The work already accomplished although fragmentary, has already proved its value and gives promise of richer results. Undoubtedly the tests at our disposal have much significance for the mathematics teacher. They have begun the detailed analysis of mathematical abilities, they have rendered possible improved methods of classification, and diagnosis and prognosis of individual ability can now be made by their help. Analysis, classification and diagnosis—these three outstanding ends they serve; and the greatest of these is diagnosis.

BIBLIOGRAPHY.

CHILDS, H. G. "The Measurement of Achievement in Algebra." Bulletin, Extension Division, Indiana University, Vol. 2, No. 6. pp. 171–183.

COLEMAN, W. H. "Scale for Testing Ability in Algebra." Superintendent of Schools, Bertrand, Nebraska.

FULLERTON, G. S., and CATTELL, J. McK. "On the Perception of Small Differences." Publications of the University of Pennsylvania Philosophical Series, No. 2, The University of Pennsylvania Press, Philadelphia, 1892.

HOTZ, H. G. "First Year Algebra Scales." Teachers College, Columbia University Contributions to Education, No. 90, New York City, 1918.

IRWIN, H. N. "A Preliminary Attempt to Devise a Test of the Ability of High School Pupils in the Mental Manipulation of Space Relations." *School Review*, Vol. 24, October, November and December.

METZLER, W. H. "Problems in the Experimental Pedagogy of Geometry." *Journal of Educational Psychology*, Vol. 3, p. 545.

MINNICK, J. H. "An Investigation of Certain Abilities Fundamental to the Study of Geometry." The University of Pennsylvania Press, Philadelphia, 1918.

MONROE, W. S. "A Test of the Attainment of First-Year High-School Students in Algebra." *School Review*, Vol. 23, pp. 159–171.

ROGERS, AGNES L. "Experimental Tests of Mathematical Ability and Their Prognostic Value." Teachers College, Columbia University, Contributions to Education, No. 89, New York, 1918.

RUGG, H. O., and CLARK, J. E. "Standardized Tests and the Improvement of Teaching in First-Year Algebra." *School Review*, Vol. 25, pp. 113–132 and 198–213.

"A Cooperative Investigation in the Testing and Experimental Teaching of First-Year Algebra." *School Review*, Vol. 25, pp. 346–349.

"The Improvement of the Ability in the Use of the Formal Operations of Algebra by Means of Formal Practice Exercises." *School Review*, Vol. 25, pp. 546–554.

"Scientific Method in the Reconstruction of Ninth-Grade Mathematics." The University of Chicago Press, Chicago, 1918. Supplementary Educational Monographs, the *School Review*, Vol. 11, No. 1.

STOCKARD, L. V., and BELL, J. Carleton. "A Preliminary Study of the Measurement of Abilities in Geometry." *Journal of Educational Psychology*, Vol. 7, p. 567.

STROMQUIST, C. E. University of Wyoming, Laramie, Wyoming.

THORNDIKE, E. L. "An Experiment in Grading Problems in Algebra." MATHEMATICS TEACHER, Vol. 6, p. 123.

GOUCHER COLLEGE,
BALTIMORE, MD.

SOME SUGGESTIONS FOR COURSES IN MATHEMATICS FOR NON-COLLEGE PREPARATORY STUDENTS.*

The question of college preparatory mathematics has been so long under discussion in all its aspects that it would appear that special attention is due the pupil who does not expect to go to college, and for whom the secondary institution is the finishing school. Valuable as are the standard courses in mathematics as given in most high schools, other material can unquestionably be substituted for at least a part of them which will be of more immediate practical use to the pupil who expects to take up his life work immediately after leaving the high or other secondary school. It is manifestly impossible to suggest courses which will be applicable to all schools, or even to all schools of a given type, due to widely varying local conditions as well as to great differences in the caliber and future prospects of the pupils. The committee has spent much time and thought upon the subject and finds it difficult to recommend a complete definite course for any school, preferring rather to offer suggestions which may be the means of inspiring our schools to improve present courses or to construct practical and useful ones for our boys and girls.

Realizing that the variety and scope of our schools is almost without limit, we shall confine our discussion to three comprehensive types, as follows: (A) The academic high school with no shops or other special mechanical equipment, (B) the high, trade, or vocational school with special mechanical equipment, and (C) the commercial school.

(A) The Academic High School with no Shops or other Special Mechanical Equipment.

This class includes the large majority of our high schools, ranging from the small school of the country town to the well

* Report presented at the spring meeting of the Association of Teachers of Mathematics in New England, May 3, 1919.

established high grade institution of the city. From their ranks come a large percentage of the pupils who are preparing for college. Of those who are undecided, or who do not expect to go to college, some will make up their minds to continue in a higher institution after having taken one or two or three years in high school. The courses in mathematics should be so arranged that these pupils should lose no more time than is necessary in making the transition from one course to the other, although it certainly would not be wise judgment to require all pupils to take full college preparatory work simply because some expect to, and others possibly may go to college. A course which is of value to a person who does not expect to attend a college should also be of value to him who carries his studies into the higher institutions. The pupils should be helped to find themselves, and those who have little or no expectation of going to college should be given the material which will be of the greatest immediate practical value to them.

Nearly every intelligent and progressive man uses arithmetic in his work to a greater or less degree. The deplorable weakness of many of our high school graduates in the use of arithmetic calls forth abundant criticism of much of the mathematics which we require of our pupils. By all means the most important thing, mathematically speaking, for our boys and girls is a thorough grounding in the principles and practice of common arithmetic.

For pupils of this class the following is recommended:

Ninth and Tenth Years.—(*a*) Two years of elementary algebra and plane geometry, in common with the college preparatory group for boys and girls alike. The question of whether a year of algebra should be followed by a year of geometry or the reverse, or whether there should be a judicious dovetailing of one into the other, is not appropriate to this particular paper.

Eleventh and Twelfth Years.—(a) A review of arithmetic, particularly the four fundamental operations as applied to whole numbers, common and decimal fractions, percentage, square and cube root, with emphasis placed on mental calculations, accurate results and short methods.

(*b*) Use of the slide rule.

(*c*) Plane and solid mensuration, with a large number of practical problems based upon it.

(*d*) Simple mechanics of forces.

(*e*) Logarithms, their theory and practical use.

(*f*) A brief, practical course in trigonometry, with emphasis on the use of tables and the solution of triangles.

(*g*) Theory and practice of simple surveying, including the use of transit and level, with taking of notes in proper form, based if possible on the student's own observations.

(*h*) Such problems in the machinist's, carpenter's and other trades as may be appropriate without special equipment, selecting local practical problems as far as possible.

(*i*) Arithmetical and other problems relating to house-furnishing, dress-making, millinery, dietetics and household economics.

Courses (*a*) and (*c*) should be for boys and girls alike, although they may be abridged for the girls if desired. Courses (*b*), (*d*), (*e*), (*f*), (*g*) and (*h*) are for boys only, while (*i*) is for girls alone.

While many academic teachers may rebel at the suggestion of such courses as have been mentioned, honest thought and consideration for the real benefit to the pupil must convince him that such material as will help the boys or girls to gain that which will be of immediate and real value to him in his work is of more importance to the person who does not go to college than some of the usual forms of pure mathematics.

(B) The High, Trade, or Vocational School with Special Mechanical Equipment.

Pupils in these schools should be given little pure mathematics that is not of direct practical value, but the greatest possible amount of the kind which will be of real use to them in the office or shop.

Arithmetic should be reviewed until the pupil is able to handle common and decimal fractions, percentage and square root with speed and accuracy. Skill should be developed in mental calculations, estimating results, and short methods for addition, subtraction, multiplication and division. Arithmetic, algebra

and geometry should be blended as far as possible, although each subject should be recognized and taught as such.

The work in algebra should be based on the idea of teaching the pupil to solve ordinary equations, write, transform and apply formulas, and work with graphs as the principal end in view. The course should include algebraic expression, simple, simultaneous, and quadratic equations with problems, and the plotting of graphs for a variety of curves and equations.

The formal work in geometry should be cut down to a small number of essential theorems. A large number of problems in plane and solid mensuration should be given, until the pupil can find dimension, surface, or volume for any common geometrical form, or object formed by combination. This work should be correlated with the study of formulas under the head of algebra.

A course in trigonometry and logarithms is important, and should by no means be omitted. It may be comparatively brief, but should be sufficiently thorough to enable the pupil to solve triangles of all kinds, including applications to practical problems.

In connection with the above, instruction should be given in the use of the surveyor's level and transit, with taking of notes in proper form and solution of triangulation problems.

Considerable time should be devoted to the subjects of shop mathematics, mechanics and strength of materials. A thorough knowledge of very elementary mathematics is sufficient to bring nearly every ordinary practical problem within the capability of the average older schoolboy. The large majority of the shop and office problems are purely arithmetical, and the boy must learn that the ability to solve them depends not so much upon his knowledge of mathematics as upon the exercise of logical reasoning. This course, or group of courses, is of great importance from the practical standpoint, and should include proportions of screw threads and gears with a variety of problems in machine-shop, carpentry and other lines of trade work, as well as problems in materials subjected to tension, compression, and shear, bending moments, investigation and design of simple beams and columns, graphical and analytical determination of stresses in simple framed structures, and design of members. Work of this kind can be amplified almost

beyond measure, according to the local conditions and demands. The pupil should by all means use and become familiar with one or more of the standard machinist's or structural handbooks and learn to read tables of all kinds and interpolate between values in order that his work in the school may be related as far as possible to the real operations of shop and office.

Possibly a word in behalf of the slide rule may be appropriate at this point. The time has long since gone by when this instrument was looked upon as a curiosity for the mathematical shark, a new-fangled notion of no practical use. Today an engineer who does not approve of the use of the slide rule is not abreast of the times. The pupil should learn to use it with accuracy for the ordinary computations, and realize its limitations as well as its possibilities.

The work indicated above is of course intended especially for boys. For girls, mathematics is too often looked upon as an unimportant and abhorred subject,—an evil inflicted upon them by an unsympathetic teacher or inflexible system. This is truly a most unfortunate illusion for the pupil, and too often a traditional sidestepping on the part of the more or less unconcerned teacher. Girls for whom the high is the finishing school should unquestionably be given a carefully planned arithmetical course, reviewing the common operations used in ordinary calculations, followed by courses in simple algebra and mensuration. After these should be offered a wide variety of problems in domestic science, household economics, housefurnishing, dietetics, dressmaking and millinery. There have been published one or two excellent books bearing on these topics, and if carefully taught by a thoughtful teacher the girls will see a value in and take a real interest in the much hated subject of mathematics.

The above paragraphs apply particularly to schools which are nearly or entirely non-college preparatory. For those institutions which offer both preparatory and non-college preparatory work it may not be possible to make the entire course in mathematics distinctive for each group. In this case if it be a four-year course, the first two years may be devoted to elementary algebra and geometry for both alike, after which as much as possible of the work outlined above may be given to the non-college preparatory group.

(C) The Commercial High School.

Judging from observations it would appear that for a large proportion of our commercial courses work in mathematics is to a considerable degree looked upon as unnecessary or unimportant. The truth is that for no class of pupil is a working knowledge of arithmetic more important than for the commercial. Ought the commercial graduate to be an automatic manipulator of typewriter and ledger, or should he be able to apply his knowledge and ingenuity to whatever common-sensible problems occur in the course of a week's work? The person who graduates from a commercial school has as great a likelihood of becoming a general office or shop assistant as he has of becoming a stenographer or bookkeeper, and for him a thorough knowledge of elementary mathematics is well-nigh indispensable.

For commercial students, the following general outline is suggested.

Ninth Year: Business Arithmetic (Required).—(*a*) Review of addition, subtraction, multiplication, and division of whole numbers, with emphasis on the importance of pupils learning the habit of checking results.

(*b*) Thorough drill in common and decimal fractions and aliquot parts.

(*c*) Percentage, with special applications to the reckoning of gain or loss on both the cost and the selling price.

(*d*) Commercial (or trade) discount.

(*e*) Simple interest; thorough drill, with special stress on the sixty day method.

(*f*) Bank discount.

(*g*) Simple formulas; numerical substitution and the solution of equations in one unknown.

(*h*) Simple graphs as applied to examples arising in business.

Tenth Year: Business Mathematics (Required).—(*a*) Review of topics (*a*) to (*f*) of ninth year.

(*b*) Accurate (or exact) interest.

(*c*) Compound interest.

(*d*) Taxes and fire insurance.

(*e*) Life insurance (studied briefly).

(*f*) Stocks and bonds (studied briefly).

(*g*) Domestic and foreign exchange.

(*h*) Plane and solid mensuration with no formal geometrical proofs. Review (*g*) of ninth year introducing formula work wherever possible.

(*i*) Practical measurements in flooring, roofing, paving, capacity, lumber, etc.

(*j*) Commercial graphs, more difficult than in ninth year.

(*k*) Practical use of logarithms.

Eleventh or Twelfth Year: Advanced Business Arithmetic and Commercial Algebra (Elective for Either Eleventh or Twelfth Year).—(*a*) At least two months work in commercial algebra.

(*b*) Mensuration continued with a few formal proofs in plane geometry.

(*c*) Logarithms, with thorough drill in their theory and practice.

(*d*) Use of the slide rule.

(*e*) Series.

(*f*) Compound interest.

(*g*) Equation of payments.

(*h*) Annuities.

(*i*) Amortization.

(*j*) Depreciation.

(*k*) Bond valuation.

(*l*) Life insurance.

While the foregoing paragraphs are more or less general in character they are the result of serious deliberation on the part of the committee which presents them. Beyond question the special work in mathematics for our non-college preparatory pupils is sadly in need of strengthening, and it is earnestly hoped that the suggestions which have been offered may interest and encourage our teachers to attempt constructive work along this line wherever further development is needed.

Respectfully submitted,

CLARENCE E. PADDOCK, *Chairman,*
HAROLD B. GARLAND,
CHARLES E. HAIGLER,
ELMER CASE,
THOMAS G. REES.

A STATISTICAL STUDY IN CORRELATION OF EFFICIENCY IN SECONDARY MATHEMATICS AND EFFICIENCY IN OTHER HIGH SCHOOL BRANCHES.

By Nelle L. Ingels.

It is the purpose of this paper to record the results obtained from an investigation concerning the correlation of efficiency—(1) in the study of mathematics and history, (2) in the study of mathematics and foreign language, and (3) in the study of history and foreign language. A similar investigation was made several years ago by Prof. H. L. Rietz and Miss Imogene Shade, "On the Correlation of Efficiency in Mathematics and Efficiency in Other Branches," in the University of Illinois.*

What is the probability that a student who does good work in one subject will do good work in another subject? This question has been a much discussed one. Some hold that the student is "born short" along certain lines and "born long" or strong along other lines.

A statistical study of this question shows that the correlation coefficient is very high. While it is true that some students are very efficient in the study of some subject or subjects and very poor in others, they are the exceptional and isolated cases; they are not the average or the rule.

The sources of information were the records of Greenville High School and the records of the Preparatory Department of Greenville College from 1908 to 1917.

In selecting the data, only those cases in which the subject had been carried simultaneously were considered, and in each case each subject had been carried for at least one year. The year's average was used in all cases.

It seems reasonable to suppose that the general conditions will be most nearly the same if the above methods be used in

* University of Illinois Bulletin, Vol. 3, No. 1.

selecting the data. It is not assumed that all teachers grade accurately; but, that in the long run, the teachers in general, are fair and accurate in their estimate of the student's value in grading.

While the sample in each case is small, it seems reasonable to believe that it is a random sample and that, with a larger popula-

TABLE I.

$$r = \frac{1}{\sigma_r \sigma_e}\left(\frac{\Sigma D_r D_e}{n} = c_r c_e\right) \pm E = 0.59 \pm 002$$

$$E = \pm \frac{0.6745(1 - r^2)}{\sqrt{n}}$$

History \ Mathematics	75	80	85	90	95	100	f	$v-g$	$f\cdot(v-g)$	$f\cdot(v-g)^2$	D_rD_e
75	15	15	5	1		6	36	−10	−360	3600	2200
80	15	33	31	9	2		90	− 5	−450	2250	1250
85	9	32	32	13	10		96	0	−810		
90	6	14	17	50	19	1	107	5	535	2675	1625
95		5	12	20	29	6	72	10	720	7200	4550
100					2		2	15	30	450	300
f	45	99	97	93	62	7	403		1285	403)16175	403)9925
									−810	40.1365	$\frac{\Sigma D_r D_e}{n}$ = 24.6279
$v-g$	−10	1			5	10	15		403)475	c_e^2 = 1.3893	$c_r c_e$ = 0.7165
									c_e = 1.1787	σ_e^2 = 38.7473	23.9114
										σ_s = 6.2247	
$f\cdot(v-g)$	−450	−495	−945	465	620	105	1190	−945	403)245	c_r = 0.6079	
$f\cdot(v-g)^2$	4500	2475		2325	6200	1575	403)17075	42.3697	c_r^2 = 0.3694	σ_r^2 = 42.0003	σ_r = 6.4807

$$r = \frac{23.9114}{6.2247 \times 6.4807} \pm E$$

$$r = 0.592 \pm 0.022$$

tion the deviation from the results obtained, will be the deviation due to random sampling. It seems reasonable to believe that the training through the eighth grade is more nearly uniform in all schools than the training in all high schools, because of general use of the course of study of the state, and because of the county examinations for the eighth grade graduation. It is, therefore, reasonable to expect a slightly higher degree of

correlation in this study because of the more nearly uniform training than was found by Prof. Henry L. Rietz* and Miss Imogene Shade in the correlation of efficiency in mathematics and efficiency in other branches in the University of Illinois.

TABLE II.

$r = 0.53 \pm 0.02$.

$r = 0.53 \pm 0.02$

Foreign Language \ Mathematics	75	80	85	90	95	100	f	$v-g$	$f\cdot(v-g)$	$f\cdot(v-g)^2$	D_rD_e
75	12	25	12	4			53	−10	− 530	5300	2250
80	18	48	33	15	3	1	118	− 5	− 590	2950	1500
85	17	40	42	33	8		140	0	−1120		
90	7	12	30	43	26		118	5	590	2950	1725
95	3	1	3	15	34	1	57	10	570	5700	3 50
100				2	2		4	15	60	900	450
f	57	126	120	112	73	2	490		1220	490)17800	490)9875
									−1120	36.3267	$\frac{\Sigma D_rD_e}{n} = 20.153$
$v-g$	−10	− 5	0	5	10	15			490) 100	$c_e^2 =$ 0.0416	$c_rc_e =$ 0.050
									$c_e = 0.204$	$\sigma_e^2 = 36.285$	20.103
$f\cdot(v-g)$	− 570	− 630	−1200	560	730	30	1320		−1200	$\sigma_e^2 =$ 6.024	
									490) 120	$c_r = 0.245$	
$f\cdot(v-g)^2$	5700	3150		2800	7300	450	490)19400		39.591		
									$c_r^2 =$ 0.060		
									$\sigma_r^2 =$ 39.531		
									$\sigma_r =$ 6.287		

$$r = \frac{20.103}{6.287 \times 6.027} \pm 0.021$$

$$r = 0.530 \pm 0.021$$

In tabulating the results a double entry table known as a correlation table was constructed. The grades were tabulated in groups which differ by five (5). The grades for example, in the 90-class are greater than 87.5 and less than or equal to 92.5

$$87.5 < 90 \leqq 92.5.$$

Table II presents the classification of 490 induividuals who studied mathematics and history at the same time and had at least one year's grade in each subject. Any number, say for example, 19, in column headed 95 and the row with 90 at the

* University of Illinois Bulletin, Vol. 3, No. 1.

left, indicates that there were nineteen students who were in the 95-class in mathematics and 90-class in history.

Table II preesnts the classification of 490 individuals who have studied mathematics and foreign language at the same time and have at least a year's grade in each subject.

Table III presents 426 individuals who have studied history and foreign language simultaneously for at least one year and have a grade in each.

TABLE III.

$r = 0.555 \pm 0.022$.

Foreign Language \ History	75	80	85	90	95	100	f	$v-g$	$f.(-v)$	$f.(v-g)^2$	D_rD_e
75	16	14	12	3			45	−10	−450	4500	2150
80	13	37	27	17	3		97	− 5	−485	2425	1000
85	7	29	36	40	16		128	0	−935		
90	1	12	23	43	20		105	5	525	2625	1725
95		4	3	13	28	2	50	10	500	5000	3550
100					1		1	15	15	225	150
f	37	37	197	116	68	2	426		1040	426)14775	426)8575
									−935	34.683	$\frac{\Sigma D_rD_e}{D} = 20.129$
									426) 105	$\sigma_e^2 = 0.061$	
$v-g$	−10	− 5	0	5	10	15			$c_e = 0.2464$	$\sigma_e^2 = 34.622$	$c_rc_e - 0.255$
										$\sigma_e = 5.884$	19.874
$f.(v-g)$	−370	−480	−850	580	680	30	1290	−850	426)440	$c_r = 1.0328$	
$f.(v-g)^2$	3700	2400		2900	6800	450	426)16250	38.146	$c_r^2 = 1.067$	$\sigma_r^2 = 37.079$	$\sigma_r = 6.089$

$$r - \frac{19.875}{5.884 \times 6.089} \pm E$$

$$r - 0.555 \pm 0.022$$

The methods used are essentially those of Karl Pearson and Francis Galton. The theory of correlation may be found in any scientific treatment of the subjects.*

The correlation coefficient r always lies between plus one and minus one.

$$1 \geqq r \geqq -1.$$

If $r = +1$ and a grade in one subject is given, the grade in

* G. Udny Yule, Introduction to the Theory of Statistics, pp. 157–209.

the other is determined; that is, if $r=+1$ and a student receives a high grade in one subject he will receive a high grade in the subject.

If $r=-1$ and a grade is given in one subject, the grade in the other is likewise determined; that is, if $r=-1$ for mathematics and foreign language and the student received a high grade in mathematics, he will, necessarily, receive a low grade in foreign language, and vice versa.

If $r=0$, there is independence. It simply means that if $r=0$, and a student receives a high mark in one branch under consideration, the wager is even as to whether he receive a high mark or a low mark in the other subject.

In each case the correlation coefficient is greater than $+0.5$

$$r>0.5.$$

We may show by comparison how high our correlation coefficients really are. Karl Pearson in his "Grammar of Science," page 458, gives the correlation coefficient of inheritance of stature of father and son to be

$$r=0.396$$

and the stature of brother and brother to be

$$r=0.391.$$

From this study it seems that we are justified in saying that there is a very high degree of correlation in efficiency in the study of mathematics and certain other secondary subjects.

In conclusion, we may say the results given by quantitative analysis differ widely from those obtained from impressions, which are formed, too often, from isolated cases.

ORDNANCE DEPARTMENT,
WASHINGTON, D. C.

A MATHEMATICAL RECREATION.

Some years ago THE MATHEMATICS TEACHER described the work an attorney in Washington had done in relation to the right triangle, in which he was interested as a recreation.

The following communication is also from an attorney, Mr. Alfred L. Booth, of Provo, Utah. It outlines some investigations he has made, not as a mathematician, but as one interested in numbers. Such recreational use of the simple facts of mathematics seems to emphasize the fascination of the subject, even to those who are not professionally engaged in its study.

As a matter of interest, the editor has appended a note showing a technical method of investigating the general case of one phase of this same subject. Others may wish to make similar investigations of other parts of it.

SOME ANGLES OF THE RIGHT TRIANGLE.

BY ALFRED L. BOOTH.

In attempting to determine: (1) Can a right triangle be geometrically constructed, having one of its angles exactly one degree? and (2) How near to an isosceles commensurable right triangle can a construction approach? the following methods were pursued.

1. Beginning with that most familiar theorem in geometry: "The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse," and taking a triangle with sides 3, 4 and 5, it will be observed that the square of the shorter leg 3 (9) is equal to the sum of the other leg and the hypotenuse (4 and 5).

The square of any odd number may in like manner be divided into two parts differing by unity. The odd number and the two parts of its square will always form the lengths of the sides of a right triangle.

As illustrations:

$$3^2 = 9 = 4 + 5; \quad 3^2 + 4^2 = 5^2.$$

$$5^2 = 25 = 12 + 13; \quad 5^2 + 12^2 = 13^2.$$

$$7^2 = 49 = 24 + 25; \quad 7^2 + 24^2 = 25^2.$$

$$9^2 = 81 = 40 + 41; \quad 9^2 + 40^2 = 41^2.$$

And so on indefinitely.

Now $113^2 = 12769 = 6384 + 6385$.

The smallest angle of a right triangle formed by lines of 113, 6,384 and 6,385 is 1° 00′ 51″.

Again; $115^2 = 13{,}225 = 6{,}612 + 6{,}613$.

The smallest angle of the right triangle with sides 115, 6,612 and 6,613 is 0° 59′ 48″.

In the one case the smallest angle is slightly greater, and in the other it is slightly less than one degree; in neither case however is it exactly 1°.

The principle can be carried on to infinity.

There being no odd number between 113 the number which gives an angle a little above 1°, and 115 which gives an angle just under 1°, the effort along that line was discontinued. In all the cases given, the sides of the respective triangles are always commensurable, and the triangles must of necessity contain in each instance a right angle.

2. This method could not be pursued in the opposite direction, that is, in trying to construct a right triangle whose legs are more nearly equal than 3 and 4. This follows from the fact that the only numbers less than 5 are 1, 2, 3 and 4 and there are no two of these except 3 and 4, the sum of whose squares will form a perfect square.

Partially by accident, and partly by trial, combined, the following sets of numbers were discovered.

A casual inspection reveals no special relationship between these numbers. It will be found by trial, however, that in each set the square of the number in column A is equal to the sum of the squares of the corresponding numbers in columns B and C. The numbers in the respective sets of Columns B and C differ by unity, and therefore if a right triangle is formed with

the numbers in Set III (29, 21 and 20) the two acute angles will be more nearly equal than they will in set II (5, 4, 3) and so in any set succeeding III the numbers being larger and differing in each case by unity, will give a right triangle approximating the isosceles, yet never reaching it exactly.

	A	B	C
I	1		
II	5	4	3
III	29	21	20
IV	169	120	119
V	985	697	696
VI	5741	4060	4059
VII	33461	23661	23660
VIII	195025	137904	137903
IX	1136689	803761	803760
X	6625109	4684660	4684659
XI	38613965	27304197	27304196
XII	225058681	159140520	199140519
XIII	1311738121	927538921	927538920

Each number in Column A is found by multiplying the preceding number by six, and subtracting from the result the number preceding the one multiplied.

Again illustrating: 5 (the number opposite II in Column A) multiplied by six equals 30; subtract 1 (the number opposite I in Column A) gives 29 the next number (III) in Column A.

6 × 29 (III Column A) equals 174; Substract 5 (II, Column A) equals 169 (IV Column A).

6 × 169 minus 29 equals 985 (V Column A).

6 × 985 minus 169 equals 5,741 (VI Column A).

And so on to any desired number of results.

Going to columns B and C.—If the sum of a pair in these columns is multiplied by 4, and the number in Column A preceding this pair is added, the result gives the next number in Column A.

Illustrations: 4 plus 3 (II Columns B and C) equals 7. Four times 7 equals 28, to which add 1 (I Column A) gives 29 (III Column A). Twenty-one plus 20 (III Columns B and C) equals 41; this multiplied by 4 gives 164; add 5 (II Column A) gives 169 (IV Column 4). And so on.

This process has been carried on until the following three

numbers were obtained (A) 3991330585377051287 29; (B) 2822296922877385246 80; (C) 2822296922877385246 79. The square of (A) equals the sum of the squares of (B) and (C).

None of the triangles formed with any of the sets of figures was such that the difference between the two acute angles was either one degree, or any multiple or fraction out of which one degree could be formed; hence the first question at the beginning of this article to date is answered in the negative; to the second question the answer is infinity.

A practical application may be made of the principle where it is desired to construct a right triangle without instruments other than a tape; or where one wishes to test the accuracy of a purported right angle. Because with carefully measured lines of a triangle with either 29, 21 and 20; 169, 120 and 119; 985, 697 and 696, one of the angles must be an exact right angle.

Going to the speculative: A right triangle having the sides before given of 3991330585377051287 29 for the hypotenuse, and 2822296922877385246 80 and 2822296922877385246 79 for the legs, supposing the hypotenuse to be the diameter of the earth's orbit around the sun, will be so nearly isosceles that the lengths of the legs will differ by less than the one-thousandth of a hairsbreadth. And the sine of each acute angle will coincide with the sine of an angle of 45° (or the square root of 0.5000000000000000) to twenty figures. By increasing the numbers, the identity can be extended to any desired length.

PROVO, UTAH.

Note.—To find the formulas for the sides of all right triangles such that the hypotenuse and one leg of each have a given constant difference. Let the legs be x and y, the hypotenuse $x+c$; then $(x+c)^2=x^2+y^2$ and $y^2=2cx+c^2$.

But $2cx+c^2$ will be a perfect square if it takes the form $2c(2cn+2cn^2)+c^2$, whatever the value of n.

$$\therefore y^2=4c^2n^2+4c^2n+c^2=c^2(2n+1)^2.$$

From which

$$y=c(2n+1)$$

$$x=2cn(n+1)$$

and

$$x+c=c(2n^2+2n+1).$$

If n takes all integral values a complete set of right triangles fulfilling the conditions will be obtained.

For the case $c = 1$, the results will be the same as those obtained in Mr. Booth's first table. It is evident from the formulas that the sides of the triangles whose sides have any other difference can be obtained by multiplying the numbers in this set by the required difference.

When $c = 1$, the resulting formulas also show the facts used by Mr. Booth in his work; for y has the general form for an odd number, $2n + 1$, and its square, $4n^2 + 4n + 1$, (y^2), is composed of two integers differing by one, $2n^2 + 2n$, (x) and $2n^2 + 2n + 1$, $(x + 1)$.

Also

$$(2n^2 + 2n)^2 + (4n^2 + 4n + 1) = (2n^2 + 2n + 1)^2$$

or

$$x^2 + y^2 = (x + 1)^2.$$

E. R. S.

A PSYCHOLOGICAL BASIS FOR A SYSTEM OF EDUCATION WITH APPLICATIONS TO MATHEMATICS.

By W. Paul Webber.

The aim of this paper is to put in small compass and in non-technical form a theory of education that can be applied directly by the serious teacher, to assist in orienting some questions that are confusing to a considerable number of teachers of mathematics, and to furnish a statement of some of the definite objects that may be accomplished in teaching mathematics. The organization of the material is incidental to a course of lectures on the theory and methods of teaching mathematics given regularly by the writer in the University of Pittsburgh. While some controversial questions are discussed, it is the purpose of the writer to be constructive.

It will be convenient to begin with a brief survey of states of mind. Let us call any thing that enters into the complexity of a state of mind an element of state of mind. These elements are obtained through the sense experience, memory and higher mind processes. Thus any thing that comes into consciousness through the senses as well as through the memory processes is an element of state of mind. The complex of these elements is variable with time. It is also variable with the condition of health and with the general mental condition of the individual.

As a function of time the complex has two increments, one negative and one positive. With a young child the complex will be relatively simple. The experiences of the developing individual furnish a large number of elements that are available later through memory. A certain number are lost through disuse or failure to make a firm impression. At no time is the available complex identical with the sum total of elements acquired to date. The complex of available elements may then be looked on as a wave in time whose content changes with time by leaving some elements in oblivion and by adding new

ones through new experiences and through transformations of old ones. The magnitude of the wave also varies, starting very small, increasing to a maximum and finally diminishing till the end of life. Some elements remain in the wave through life, or the greater part of it.

One primary function of mind is its capacity of choosing or determining at any time what element shall enter consciousness or receive attention. This implies what in common parlance might be called deliberative or voluntary control of attention. Individuals vary greatly in this power. There is of course the non-voluntary aspect of attention which will not be taken up in this discussion. Undoubtedly proper exercise and instruction will strengthen and direct this voluntary control of attention to useful purposes. This is one of the main objects of education in a broad sense. A mind with this primary function well developed affords great possibilities. That the control of the sources of the complex of elements that make up states of mind must affect this primary function is shown in the results of environment, social activities and the public press on the citizens of a state. The acquisition of the proper kind of elements as a means of intelligent direction of the mind will be a powerful factor in education.

Two fundamental principles will now be stated.

(1) Every state of mind tends to arouse action, advantageous or possibly neutral.

(2) The kind of reaction is determined by the state of mind immediately preceding it.

From these principles it is evident that idleness is unnatural and that to secure some direction and control of the conditions that furnish the complex of elements that make up the states of mind is desirable. Without this control much waste and even harm may result. Further, there must be some source of elements that will tend toward a right and healthy basis for judging the values of the states of mind. For harmful experiences may be pleasing, while right experiences may be less pleasing,—though not necessarily repulsive,—and consequently less likely to call forth advantageous action. This source must be furnished somewhere in the sum total of the educational means used with the individual. For example the reading to

excess of light trashy (not immoral) fiction stupefies the mind. The mind atrophies from lack of sufficient effort to obtain entertainment. This result has been noticed by many teachers. There must be some control over the environment to the end that the individual shall learn to secure some of his entertainment from other and more stimulating sources until he reaches judgment that can be trusted to direct him. This amounts to the formation of habits sufficiently fixed to direct his actions.

Habit may be defined as follows: When an act (of body or mind) has been repeated so often that the voluntary effort required to perform it has been noticeably reduced below that required for the first performance, habit is said to exist with reference to that act. Habits are classified as habits of body movement, and habits of association of the elements of state of mind, that is, of thinking. Habits are essential to efficient thinking as well as to movements. Habits limit the variety of choice of the individual, but nevertheless a variety of habits is necessary to readiness and efficiency in life's actions. Few terms used by the moral lecturer and the psychologist are more misunderstood than habit.

The layman almost invariably thinks of habit as a condition under which an act is all but inevitable from having been repeated many times; that the act is outside the control of the mind. By the psychologist the term is used in quite a different sense, as the above definition shows. Habits are of all degrees of fixedness, from the merest probability of the supposed action, to the strongest presumption of the performance of the act.

Habits in which the voluntary effort required to perform the act is very little reduced below that required for the first performance are much less likely to bring forth the habitual action than habits which are more strongly cultivated. A person with a number of habits weakly fixed cannot be counted on for positive action as can one with some habits strongly cultivated. The first may have a pliability that can be put to good purpose, but he is apt to be only a drifter. The second can be counted on for positive action under given conditions. He may be less pliable, but is hardly so likely to be a drifter as the first. He is therefore in general regarded as a better citizen, assuming of course that his habits are not evil.

Practice must in general be imposed to secure any desired habit. Hence an important problem of education is to find ways and means to secure the formation of desired habits. A considerable degree of fixedness of habits is in general necessary to ensure prompt and efficient action. A choice must be made from the maze of possible habits that can be established. It is often difficult to determine what and how many habits should be highly cultivated to secure the best results. The number of habits that one can efficiently cultivate is limited. That one single habit would be but a risky preparation for life in a complex society must be evident. Some variety is necessary and desirable. Train a person in a single set of mechanical movements, necessary to do a certain thing, say drive a nail, or train him to great skill in the ordinary addition table. It is conceivable that a person with either of these as his sole equipment, above his natural physical habits, might be able to earn an existence. He would have no option as to vocation. His life is nearly closed to opportunity from the start. No one would say he is educated in any adequate sense.

Education is defined as a process of forming habits of: (1) Observation (use of the sense organs), (2) thinking, (3) movements of the body. This definition seems to offer broad opportunities for study. Education is obtained through instruction, training and experience of the general and elemental or fundamental activities of body and mind to form habits that will ensure: (1) Character in the ordinary acceptation of that term, (2) efficiency or skill in movements and thinking with reference to the general aspects of life, (3) preparation for some field of productivity, the main field, (4) capability for the highest proper pleasures of life, (5) sympathy and helpfulness toward fellows, (6) desire to promote race welfare and betterment.

A scheme of education that does not provide for all of these is deficient to that extent. Of course much of this educational scheme will lie outside the formal school system. Its effect is none the less certain however.

Passing now to the consideration of some accepted principles of training, the following may be mentioned:

1. A definite relation exists between skill in one field and general intelligence in that field. The general intelligence can

furnish perspective, sense of proportion and balance for the technical skill.

2. The best way to attain skill in any field is to train in that field. The best way to learn to play foot ball is to go into the game on the training field for that purpose. It is hard for one to form an idea of what would be the result, if it were possible to train a candidate for foot ball who had no other knowledge or training at all. Practically no such candidates exist. The relation that exists between their previous knowledge and their progress in foot ball would be interesting. Possibly something is known on this point that would be useful if made general knowledge. This illustration is only a typical one, however. What are the essentials of a course for electrical engineers? What general knowledge in the field of electrical engineering is desirable? What relation exists between this general intelligence and the work of the engineer in its practical aspects?

3. General training is essential to a full understanding of any thing, for by many things is one thing known, that is by apperception.

4. In general the highest skill or success in any field is assured only when training is had in all related fields.

Here is an important and difficult problem, viz., To determine all the fields related to a given field and to determine whether or not the relation is an essential one. No doubt there is some useful material on this point among the experiences of teachers if it could only be collected and made available to the public.

5. On the other hand, training in related fields does not guarantee success in the main field, but should normally be contributory to that success. This may seem at first to contradict the last condition, but it is not to be so interpreted.

This brings us to the question of transfer of training. It is generally admitted that some transfer takes place. Its measure may however be positive, negative or in some cases neutral as previously indicated. That transfer will take place automatically without any care on the part of the teacher is not claimed by the writer. That it often does take place under such conditions is probably true. Two aspects of transfer of training may be stated. (1) The identical element theory at one time promised to solve the problem finally. It teaches that transfer

depends on a common function or action that operates in two or more fields. According to this theory knowledge of the addition table would be transferrible to accounting because adding is a part of accounting. This is a rather specialized example. Writers recognize attention, control, accuracy, caution, etc., as possible transferrible habits or attitudes. The identical element theory has not come up to its original promise. It has been useful in promoting research and in suggesting extensions, but it is now conceded to be inadequate. Some very definite general habits have been shown to be transferable. A few such are inserted here from a paper by Professor Angier published several years ago. (1) Attitude or habit of verification—asking for evidence instead of reasoning alone from assumptions or first principles. (2) Problem attitude or vital attitude. (3) Attitude of satisfaction with accomplishment. (4) Attitude of correlating theory with applications.

To attain the first, pupils should be taught to reason from first principles, but they should also be taught to seek external evidence and to verify by observation and experience. Training for this is as necessary as for other objects of education. The second indicates that all who are capable should be taught to approach new situations as problems and not as situations on which authority alone is to be regarded. Teach the student to doubt, to question, to weigh, and to seek a solution to his problems. Teach so as to cultivate this attitude. The third is a refined but proper attitude and a useful one, if it can be attained. It should encourage doing things well and doing things worth while. It should oppose slovenliness and stimulate ambition for better things. The fourth is opposed to formalism in mechanical drill. It should favor keeping in touch with affairs. It can be cultivated only by persistent practice. It favors transfer in particular fields.

Consider a person trained only to drive a nail. What else might this training help him to do? What other work calls for the same set of movements? What partial set of movements can be separated and used in some other work? Consider chopping with a hatchet. The same grip of the handle, the same swing of the arm seem to be common to the two. Chopping will of course require some new movements. Can we say a

person will chop more readily after having learned to drive a nail? Probably. Does the transfer depend alone on the transfer or a set of mechanical movements? Probably not.

Consider a person trained only in the addition table. What other work will this help him to do? Book-keeping? More adding. Suppose you are just beginning to teach multiplication. What previous knowledge is it customary to employ? Addition? Probably so. A knowledge of addition is used to discover a new relation among numbers and leads to a new process in important respects quite different from the old one. Is this transfer? Probably so. Does this transfer depend alone on the transfer of a set of mental processes? Probably not. These illustrations imply at least one more element to be considered in the transfer of training. Some important mental process doubtless takes place. The nail driver recognized that the movements of chopping are similar to the movements of driving a nail. This implies a generalization. Generalization must play an important rôle in transfer. The addition table is generalized to include the larger problems of accounting. What mental act clears up the mystery of the slide rule when the pupil has been told that the slide rule is a mechanical table of logarithms?

(2) The generalization aspect of transfer teaches that generalization, by favoring extensions to application (favors transfer). It is probably the greatest instrument of the mind for mastering new situations and extending knowledge. Generalization is the highest aim of instruction. Formalism does not reside so much in subjects as in methods of presentation and study. This applies with more force to elementary and secondary instruction than to more advanced instruction. For it is with the former that the desired habits must be started rather than in the latter. It is probable that generalization is more fundamental in transfer than identical elements in that generalization is the means of pointing out the identical elements.

Some inference may be drawn from what has preceded. Any group of ideas, principles or training processes for educational purposes should: (1) Contribute to the formation of correct habits of personal character, (2) contribute training directly usable in the main field, (3) contribute toward a general under-

standing of things, especially in the main field, (4) contribute traning transferable to the main field or to a possible new field, (5) be capable of organization for systematic and serious presentation.

Resuming the fourth, there are certain conditions of teaching that favor transfer of training: (1) In teaching emphasize the method of the subject as well as the subject matter, (2) by illustrative examples, show how the method of the subject may be extended to other fields and to problems, (3) develop the subject in a fashion somewhat similar to that actually experienced by the race or in use by investigators, (4) the steps in the development should as far as possible be kept in full view of the pupil, (5) the pupil should as far as possible be led to make the adjustments the subject is intended to enable him to make.

This paragraph specifically raises the question that the teacher must so direct his teaching that he will attempt, at least, to realize for the pupil the transfer values of the subject. This question many teachers dislike to consider. A few may even refuse to consider it. The chief consideration with these seems to be to attain a high training in the subject per se. This may be right in dealing with mature students who have already formed the habits which make transfer possible and who are seeking skill for a particular purpose. It is recognized that the transfer value of a subject does not always follow immediately after the instruction, but sometimes years after. This seems to render uncertain the findings of experiments conducted for a few weeks and then tested immediately for ultimate transfer values. However, this fact must not be used to justify neglect in teaching for transfer values on the plea that the transfer will come at some more remote time anyway. If the above principles are observed carefully there will be more noticeable transfer both in short periods and in longer ones than if no care be taken in this regard.

Some examples illustrating the use of the preceding theory may be of interest. Consider the case of elementary algebra. One writer has laid down four main aims for subjects in teaching elementary algebra. (1) Analysis, (2) direct use of symbolism, (3) extended use of symbolism, (4) manipulation of

symbols. It seems obvious that these four processes have wide use in intellectual affairs and in applications to practical affairs. These values will not occur to the average pupil unless pointed out to him. If he is conscious of acquiring something of value he will normally be more interested in the study.

Consider the very elementary example which is typical of many of more advanced character. Suppose the child has learned that the area of a figure is the number of unit squares that will just cover it. Ask him to find the area of a rectangle 7 in. by 5 in. He will ultimately discover that five rows of seven each will cover the figure. He can be easily led to express this as $7 \times 5 = 35$ square inches. This is a particular case. Now call his attention to the fact that the only process involved in the calculation is multiplication of the units of length by the units of breadth. This is the result of an analysis of the process. The dimensions are now not particular numbers alone but general names for certain parts of the figure as well. State the above equation using the names instead of their number values. The resulting relation holds for all rectangles. This is a generalization. The process of a particular case is transferred to the entire class of similar cases.

Care should be exercised to avoid generalizing on insufficient analysis. Mere multiplicity of cases without analysis will probably leave the pupil unable to make the generalization. A very few cases well analyzed will ensure the generalization. One important part of the solution of original problems is their classification. To classify a problem the pupil must analyze it carefully and determine the class to which it belongs. This calls for analysis and generalization. When this is done the technique of the class can be transferred to the particular case in hand.

As an example of the direct use of symbolism consider the three side rule for finding the area of a triangle: From half the sum of the sides subtract each side separately. Multiply the half sum and the three remainders together and extract the square root of the product. The result will be the area. Compare this with the symbolic statement of the same rule,

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, c are the sides and s the half sum.

No mystic difficulty attends this formula if the pupil is taken into confidence and told that the symbolic form is only a linguistic transformation of the first statement, and that it is shorter and consequently economical. Give him time to see that the two forms of expression convey one and the same idea.

The method of symbols can be applied to various cases until it is well understood. Symbolic forms are often more definite and easier to grasp than the ordinary verbal expressions of the same ideas. It is easier to move forward in an argument if the steps can be expressed in symbolic form. Compare an old text on geometry, say Loomis's or Tappan's with a modern one. Which is easier to read? In what does the chief difference consist? Consider the value of symbols in a science like chemistry. This science has an algebra of its own, so to speak. No complicated analysis can proceed far without the use of symbols. It is of course to be understood that symbols do not always represent number values. The absence of material content in numbers carries over to their symbolic representatives with even stronger force. This fact gives elementary algebra an advantage of simplicity in teaching the beginning of the use of symbols and symbolic language. This statement takes for granted that the teaching will be more than the traditional kind.

As an example of the extended use of symbolism suppose the pupil has learned the index law $a^{m+n} = a^m \cdot a^n$ where m and n are positive integers. Later this symbolism will be extended to include cases where m and n are fractional or negative numbers. This is done by a further generalization of the meaning of symbols. It is suggested by the manifest convenience provided a consistent interpretation can be assigned. This is a characteristic method in mathematics. The extension of the above formula for the area of a rectangle to cases where the dimensions are fractional is a similar generalization. In each case the old process in a restricted field is carried over or transferred to a more extended field by a generalization. The formal technique is transferred bodily.

Consider still another example, $(8+5)(8-5) = 64+40-40-25$. This is evidently $64-25 = 8^2-5^2$. An analysis of the process will show that the form of the result is not dependent on the particular numbers chosen. The rule is there-

fore general. In symbols this is equivalent to

$$(a+b)(a-b)=a^2+ab-ab-b^2=a^2-b^2,$$

no matter what values are assigned to the symbols a and b. Thus by manipulating symbols we arrive at the same conclusion as by analyzing a particular case and generalizing. By manipulating symbols according to some predetermined plan we may arrive at new information. The well known law of electric batteries is expressed as

$$C=nE/(nr+R).$$

Now by manipulating the letters (symbols) as numbers we can discover two new and not immediately obvious relations which amount to new information, viz.,

$$r=E/C-R/n \text{ and } R=n(E/C-r).$$

This is a very simple illustration of the way in which a mathematician, ignorant of electricity may discover new information for the use of the electrician who may be equally ignorant of mathematics. Discoveries of this kind are of frequent occurrence in the history of science.

In this connection it may be noted that by symbolizing the unknown in an equation or a problem we secure all the advantage of knowing it so far as manipulation and calculation are concerned. This is by virtue of the manipulation of symbols. If the pupil acquires an intelligent understanding of the four mathematical instruments above named and a fair facility in technique he will possess an equipment that will be of broad use to him in later years. If too large portion of time is devoted to technique he will fail to transfer his knowledge. If he is not fairly well trained in technique his calculations will be slow and clumsy so that he will avoid their use on all possible occasions.

As part of the manipulation of symbols comes the various transformations that are used in the solution of the problem. The pupil should be taught that the equation is an instrument carrying with it a straightforward procedure for separating the unknown in it from the known. In a large class of problems the first step is to analyze the conditions with the purpose of

obtaining the equation which can afterward be manipulated by pre-established processes in such a way as to separate the unknown and determine its value. Here again we have three processes, viz., analysis, symbolization and manipulation of symbols.

It comes in the sphere of algebra to teach the graph and its uses. Statistical graphs are of frequent occurrence in various scientific and semi-scientific articles in the magazines. Almost every intelligent citizen will have occasional use for some knowl edge of graphs.

Every statistical graph pictures a relation between number pairs. The same is true of the graph of an equation. Through these ideas first notions of functionality can be taught. The first efforts in function study will be more or less crude. Not until the pupil reaches some maturity will he appreciate the significance of the function idea in its full sense, but he will be acquiring some notions of a very useful instrument of thought. A graph can express more in small compass than words and do it more vividly.

Consider still another illustration. The pupil has learned the processes of solving plane triangles in trigonometry as a purely geometric problem. He can apply the sine law or the cosine law to any triangle, having given the requisite parts. Does this knowledge ensure that he will be able to easily use it for solving the triangles of statics or astronomy? Hardly. He must learn to recognize in the triangles of statics and astronomy the same relations of parts that exist in the academic or geometric triangles he has learned to solve. He must analyze and generalize. When he has recognized the identity of the triangles in the different fields he will easily transfer his fomulas and their attending processes to the triangles of statics and astronomy. That learning to solve by the use of formulas such academic problems as: Given $a=55$, $b=75$, $c=40°$, to find the other parts—is not identical with the best use of the same formulas to find the resultant of a set of forces or to find the diameter of the sun having given the distance and parallax, few teachers can fail to observe. The last call for analysis and generalization of a more searching type than the first. They also call for an attitude of passing from theory to applications and conversely. When the

pupil has recognized the academic triangles as an instrument that can be transferred through generalization to all triangles in all situations he is well on the road to success in the solution of the problems of statics and astronomy.

We have set forth some examples of transfer, mostly through generalization, from one state or phase of mathematics to another higher and broader and of transfer of academic knowledge of a particular method to its applications through analysis and generalization. In all cases some symbolization and manipulation of symbols were necessary to the carrying out of the solutions. Some well defined attitudes which are of use to people in general and which can be cultivated in the teaching of almost any branch have been stated. Some fundamental processes and principles of algebra that are of direct use in the sciences and industrial arts and in the field of intelligent citizenship have been cited. Mathematics on account of its methods and subject matter seems to offer peculiarly favorable opportunities for instilling some of these attitudes, *e. g.*, problem attitude, verification attitude and attitude of correlation of theory with applications. Besides these several more specifically mathematical values were pointed out.

It is the firm belief of the writer that if the principles herein laid down could by the coöperation of teachers and school officials be given more attention our schools would not be criticized so much as now and we as teachers of mathematics would be better understood and the service of mathematics in a scheme of education and to civilization would be more appreciated in general. My appeal is to my fellows in the field of mathematics that they may make further study of the questions involved in order that fewer of them may assume a passive attitude toward the transformations that are in progress. For just in proportion to the part we take in the changing state of education can we command a hearing in the readjustments that are coming.

Finally a few psychological principles that have direct application in teaching mathematics will be stated.

To understand and to get a working knowledge of a definition, perception (gathering elements), analysis, interpretation and generalization are necessary.

The mental processes of learning a theory are different from

those of applying the theory. This is related to the attitude of correlation above. So called real problems are much used for this purpose. They should be carefully analyzed, classified and solved in order to realize the intended values. Solution for numerical results alone is not sufficient.

The type of attention required to follow a demonstration is higher than that required to master a definition.

Psychology cannot teach us how to teach but can and must furnish the basis on which we are to build a theory and practice of teaching. Teachers of the different branches must make contributions from their experience.

There exists (theoretically at least) an order and environment under which the teaching of mathematics (or any branch) should yield a maximum of desirable results. These conditions are not necessarily the same for all branches.

It is the business of educational science to discover the ideal order and environment for teaching purposes and to discover the adjustments and deviations of a general course of procedure in teaching to meet the different school groups.

There exist a utilitarian, a psychological and a logical basis for a theory of presentation. These must be blended to the highest advantage of the individual.

Some emphasis has been placed on the idea of transfer in this paper. This is an effort on the part of the writer to supply what he has felt to be a need. So many teachers appear to be either confused or in a state of hopeless resignation to a fate assigned to mathematics by those who are to all appearance not familiar with the subject. The writer hopes to give a basis of inspiration to a better attitude.*

UNIVERSITY OF PITTSBURGH,
PITTSBURGH, PA.

* In the preparation of this paper the writer has consulted various texts on pyschology, and in particular Judd's "Psychology of the Secondary School Subjects," Nunn's "Teaching of Algebra" and various journals of education. He desires to acknowledge his indebtedness to Professor A. M. Snyder, of this university, for valuable advice and suggestions relating to matters in pychology.

PHILOSOPHY AND NON-EUCLIDEAN GEOMETRY.

By F. A. Foraker.

Leibnitz and Descartes made remarkable contributions to both mathematics and philosophy. Newton obtains a high rank in the history of the former subject, but only a minor place in the history of philosophy, while Kant, who possessed a well-founded knowledge of the science and mathematics of his time, receives one of the foremost positions in the history of philosophy. Upon the basis of these facts, if we neglect a few of the lesser lights, the statement is often made that there is a relationship between the study of mathematics and the study of philosophy, and that he who studies one of them will also find himself a devotee in the pursuit of the other.

This is certainly not true in general of modern mathematicians and philosophers. The field covered by each of these groups has become so broad that an intimate knowledge of the other is not so easily obtained as in the days when Leibnitz wrote his calculus and his monadology.

The philosopher searches for "a comprehensive veiw of nature, an attempt at a universal explanation of things" in the words of Weber,* but in his explanations of things mathematical, at least, he has often shown that he is not familiar with the thing to be explained. Non-Euclidean geometry has crumpled many philosophical theories, but the philosopher is seldom aware of it, because he has very little knowledge, if any, of this branch of mathematics. Some of them write as if they were in entire ignorance of the existence of such a subject, others have mentioned it or commented upon it in such a manner as to show that they comprehended very little of its meaning.

Another example of this fact has recently been set forth in a work entitled "Psychology of High School Subjects."† An extract follows:

* "History of Philosophy."

† Charles Hubbard Judd.

"There is no difficulty whatsoever in talking about parallel lines that meet. In the world of complete abstraction, that is, in the world of thought about lines, we may think of lines in any way that seems to us to be desirable; but we cannot take the results of such thought back into the world of practice or even into the world of direct imagery. Anyone who, after speculation for a time about different kinds of possible space, *should try to lay railroad rails, on the assumption that parallel lines meet, would find himself in difficulty.* Speculation and practical life thus frequently need to be distinguished from each other. Indeed, the distinction is so evident that perhaps the better statement of the case is to say that the two forms of experience need reconciliation."

The mathematician may very well agree with the statement that abstract thinking leads us into regions where we should endeavor to find reconciliation with the world of reality, or he might even go further, and say that there is no assurance that we shall always find realities corresponding to the results of abstract thinking, but the author mentioned has chosen to illustrate his meaning by a very unfortunate example.

The implication of the passage from which this extract is taken seems to be that railroad rails can only be laid on the Euclidean hypothesis of parallel lines. If we are going to seek for correspondences between the results of our abstract thinking and the realities of the world, we must be sure that we get as near as possible to what those realities are, or we may write into our philosophy some errors which will be as serious as any we might retain if we did not attempt to secure such a correspondence.

The fact concerning the laying of railroad rails is that it is impossible to lay them by the Euclidean hypothesis. The author quoted talks as if he believed that in laying rails, a separate line is surveyed for each rail. Now, that is not very close to reality. Even if the earth were a plane and if the Euclidean hypothesis were demonstratively true, it would still remain impossible because instruments and men are not without error. If we were to grant the possibility of perfect instruments and the ability of men to handle them without error, still Euclidean geometry could not lay the rails, for the earth is spheroidal, and any two

straight lines that Euclidean geometry could lay on such a surface must intersect twice. If the line for one of the rails is straight and the line of the second is everywhere equidistant from it, then the second line is not a straight one *but non-Euclidean geometry can lay an equidistant line as well as Euclidean geometry.*

When we lay the rails of a railroad, a measuring bar of constant length is continually slid along between the rails as they are put down, and the problem is not one in Euclidean geometry, but the manner of actually placing the rails shows that this process is independent of any hypothesis concerning parallel lines. Moreover, it is evident that a mathematician would not use the phrase taken from this quotation, "in talking about *parallel lines that meet*" when he speaks of non-Euclidean geometry. The mathematician might say that parallel lines are *lines that do not meet* or that parallel lines are lines that meet in infinity, but even the latter definition could not enter into the method of laying rails for a railroad on a finite earth.

This recalls a quotation from William James's chapter on Necessary Truths. "If we mean by a parallel a line that will never meet a second line; and if we have one such line drawn through a point, any new line drawn through that point which does not coalesce with the first must be inclined to it, *and if inclined to it, must approach the second, i. e.,* cease to be parallel with it. No properties of outlying space need come in here; only a definite conception of uniform direction, and constancy in sticking to one's point."

Let us neglect the last clause which probably never had any definite meaning, even in the mind of James. It is evident from the glaring fallacy given here in italics that one who discusses parallelism between mental and cerebral phenomena is not necessarily competent to discuss the parallelism of lines. This is, perhaps, one of Kant's original intuitions *a priori* to all study of geometry of the last century. Do philosophers study the mathematics of the present time, or are they content to quote Aristotle and St. Augustine on this branch of human knowledge? Do mathematicians do better with philosophy?

UNIVERSITY OF PITTSBURGH,
PITTSBURGH, PA.

MATHEMATICAL TESTS:—THEIR RELATION TO THE MATHEMATICS TEACHER.

By J. H. Minnick.

To-day we are hearing a great deal about standardized tests and scales. School surveys make extensive use of them in determining the efficiency of school systems, and bureaus of educational measurements are being established throughout the country for the purpose of applying them systematically to our schools. Such a movement can not fail to make its influence felt in the class room. The purpose of this paper is to discuss mathematical tests in their relation to the class-room teacher. The development of tests and their use for administrative and experimental purposes concern us only in so far as they affect the teacher of mathematics.

We shall first discuss some of the dangers which attend the use of tests and scales. Later we shall consider the advantages which result from their use.

The most successful mathematical scales and tests can be developed only by taking into account the contributions which mathematics should make to the realization of the general aim of education. Education does not, or at least should not, consist of independent factors each contributed by a specific subject. The contributions of the various subjects should fit into each other to form a unit. The aim of education should be definitely stated and each subject so taught that it will contribute most effectively to the realization of this aim. If then a mathematical test is to serve the purpose of education most effectively, we must be sure that it measures an ability which is essential to the aim of education. Furthermore, we must have a means of measuring every such essential ability or our subject will be in danger of making a stinted contribution to education.

Here then is one of the chief dangers connected with the use of mathematical tests. Either the makers of mathematical tests have not been conscious of the educational value of mathe-

matics, or they have been unable to develop tests for the measurement of all abilities underlying that value. Most tests deal with the mechanical or tool phases of the subject and fail to measure the more important products of mathematical education. It is true that in some cases* reasoning tests have been developed both in algebra and geometry; however, the exercises of these tests do not consist of out-of-school situations such as those to which the child will have to apply his mathematics in the future. The ability to do these tests does not imply the ability to use mathematics in the solution of real life situations. Moreover, I know of no tests that attempt to measure the child's appreciation of the relation of mathematics to his material and social environments, nor have I found an attempt to measure the child's ability to recognize a mathematical situation or his ability to use his mathematical knowledge to improve present methods of work in the shop, office, or elsewhere.

Although existing tests are thus only a partial measure of the educational products of mathematics, nevertheless superintendents and administrators are using them either consciously or unconsciously as a measure of teaching ability. If pupils of one teacher make a higher score than those of another, there is a feeling that the former is the better teacher, regardless of the unmeasured results obtained by the second teacher. If thus judged, the teacher naturally gives attention to those phrases of the subject in which her pupils are to be tested, which of course means that she will emphasize the mechanical to the neglect of the more important values of the subject. Hence, unless tests and scales are used with the greatest care there is danger of formalizing mathematics more than it ever has been and thus defeating the purpose of that better group of teachers who have been trying to vitalize the work of our secondary schools.

If, as we shall note later, a scale is to be used for the purpose of diagnosis and for determining when a given ability has been sufficiently developed, then it should, throughout all its parts,

* H. G. Hotz, "First Year Algebra Scales," pp. 8–9, 18–20. Agnes L. Rogers, "Experimental Tests of Mathematical Ability and their Prognostic Value," pp. 21, 29–31, 39–40. Indiana University Algebra Tests, Tests VII and XII. J. H. Minnick, "An Investigation of Certain Abilities Fundamental to the Study of Geometry," pp. 15–19.

measure the same ability. This ability may be dependent upon others in such a way that the latter are indirectly measured; but it is important that in passing over any two parts of the scale the ability directly involved shall vary only in quantity and not in quality. It is doubtful if this principle has been observed in the construction of certain tests. In Woody's Addition Scale* of the two exercises $\begin{matrix}23\\18\end{matrix}$ and $25+42$ the latter is assigned the greater value. Is it true that the solution of the second exercise involves exactly the same kind of ability as the first but to a greater degree? Is there no new difficulty involved in the second, calling into play a new type of ability? Must a child be able to solve satisfactorily problems of the second type before he can satisfactorily solve the first type? Again in the same scale we find three exercises arranged according to their relative difficulties as follows:

(33)	(34)	(35)
.49	$1/6+3/8$	2 ft. 6 in.
.28		3 ft. 5 in.
.63		4 ft. 9 in.
.95		
1.69		
.22		
.33		
.36		
1.01		
.56		
.88		
.75		
.56		
1.10		
.18		
.56		

If a pupil does (33), (34) and (35) correctly, does it mean that in each succeeding case he is exercising the same ability only to a greater degree; or if he does (33) but can not do (34) or (35), does it mean that (33) marks the limit of his ability to

* Clifford Woody, "Measurements of Some Achievements in Arithmetic," pp. 4 and 16.

do problems of that type and that he should therefore have further drill on the addition of decimal fractions? Dr. G. H. Hotz* has developed one scale for addition and subtraction, another for multiplication and division, and a third for equations and formulæ. Do the two subjects composing each of these pairs involve the same ability? Do the solutions of $4r+3r+2r$ and $8c-(-6-3c)$ require the same type of ability? Does it require just a little more of the same kind of ability to pass from

$$\frac{p^2+4p-45}{p^2-2p-4}\cdot\frac{p^3-8}{p^2-81}\cdot\frac{1}{3pr-152}$$

to

$$\frac{X^2+27}{X^2-X-12}\div\frac{3X+9}{X+4}?$$

Will any amount of increase in the ability to solve

$$c-2(3-4c)=12$$

enable the pupil to solve the following:

The area of a triangle $=\frac{1}{2}\,bh$, in which $b=$ length of the base and $h=$ height of the triangle.

How many square feet are there in the area of a triangle whose base is 10 feet, and whose height is 8 feet?

Unless these questions can be answered in the affirmative, such tests, although having other important functions, can not safely be used for the purpose of diagnosis and for determining when a class has gained a satisfactory mastery of a given phase of the work.

We have referred to the tendency on the part of some to use tests as a means of comparing teaching ability. It should be definitely understood that generally neither absolute nor relative teaching ability can be determined by the use of the tests. In the first place, the norms for the various tests have been established by examining large numbers of children in our schools. Hence, these norms represent conditions which actually exist rather than those which should exist. We freely admit the inefficiency of our present school system and there-

* G. H. Hotz, "First Year Algebra Scales," pp. 5-7.

fore it is probable that these norms are not correct. Doubtless in many cases they are too low and in some cases they may be too high. For example, in algebra it may be that we devote so much time to long division, factoring, complicated fractional equations and the like, that either the degree of skill attained passes the point of usefulness or we are forced to neglect other topics which would have a greater usefulness than that secured by excessive drill on these subjects. If such conditions exist, a norm based upon them would be too high. Hence, since we do not know that the standards established for these tests are correct, it is impossible to use them as an absolute measure of a teacher's ability. Furthermore, the achievements of pupils are functions of many variables, chief among which are teaching ability, methods of instruction, standard school conditions, home influence, and the pupils' native ability. No one can say just what each of these factors should be in order to maintain a certain standard of achievement. Until this can be done we can not say to what factor a low score is due, and therefore, even if we knew that the established norms are correct, tests could not be used as an absolute measure of teaching ability.

In like manner the relative abilities of two teachers generally cannot be determined by the use of tests. To do this, the scores made by the pupils of the teachers in question would be compared. But since these scores are the results of many factors, a variation in them may be due to a variation of some factor other than teaching ability. Hence, it is impossible to compare the teaching abilities of two teachers by means of tests unless all other factors are constant, a condition which seldom or never exists.

It is highly desirable that tests or scales for measuring individual progress and for diagnosing the weakness of individual pupils should be developed. Such an impersonal standard for judging individuals would be of value in rating pupils and such a means of diagnosis would render individual instruction far more efficient. However, a pupil's reaction towards a test is conditioned by his physical condition, his mental attitude created by the test, his individual store of knowledge, and various external influences. The influence of these varying factors may be partially eliminated by the use of several equivalent tests

given at different times. But such equivalent tests have not been developed and until they have been, too much emphasis should not be placed on the scores of individual pupils. This difficulty can, however, be partially removed and existing tests be used with individuals to advantage if the teacher will check the results with supplementary tests of her own making.

So far we have considered the dangers that attend the use of tests. Nevertheless, when properly constructed and used, they should prove to be of great value to the class-room teacher. In the first place, we should realize that the tool side of mathematics is essential to the more valuable phases of the subject and we should welcome any means of measuring growth along these lines. As indicated above, it is for exactly this purpose that most of the existing tests have been developed. These tests may serve at least three important functions. In each of these cases they will be used to test groups of children and the large number of pupils will tend to eliminate the individual variations which are so troublesome when we are concerned with individual scores.

First, they may be used as an impersonal standard by which a teacher may measure the group products of her efforts. Although the established norms are based on conditions as they now exist rather than as they should be, nevertheless since they combine the judgment of many teachers, they are doubtless a safer standard than the opinion of any one teacher. Also the teacher's individual judgment is colored by her personal experience with her class, her likes and dislikes, and her individual ideals. Tests furnish an external means of judging, from which these personal elements have been eliminated.

Second, efficient teaching demands some means of determining when a class has attained a satisfactory standard. Unless this is possible, the teacher may pass from one phase of the work to another too soon, with a loss of time due to the fact that the pupils are not sufficiently prepared for the advanced work, or on the other hand, he may waste quite as much time by dwelling on a topic after the necessary standard has been reached. Standardized tests, properly developed, would thus enable the teacher to better economize his time. This implies that he knows exactly what ability each test measures and also

that he knows and does not neglect those other abilities for which tests have not been developed.

Finally, tests and scales may be used for the purpose of diagnosis. Teachers frequently find that their class is doing poor work, but are unable to determine the exact difficulty. If a complete analysis were made of the abilities involved in the study of a given subject such as geometry and a test were developed for the measurement of each of these abilities, then by an application of these tests the teacher could determine the exact cause of the failure and remove the difficulty with a greater degree of efficiency. Obviously, no such analysis has been made and satisfactory tests have not been developed for all known essential abilities; but a start has been made and we should not hesitate to use any test which promises aid in the diagnosis of class weaknesses. However, we must remember that tests have not been developed for all abilities necessary to the study of mathematics, and that difficulties may exist which none of the tests thus far developed will reveal.

In conclusion, it is evident that scale building is far from complete. More work has been done in mathematics than in any other secondary field. Yet, there is much to be done before we have a complete and satisfactory set of scales for our own subject. Those scales which have been developed have many imperfections and for the most part they deal with the formal aspect of mathematics. However, they constitute a step in the right direction, and although they are only a partial measure of the contribution which mathematics should make to education, nevertheless, when properly used their value to the class-room teacher can scarcely be overestimated.

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PA.

BOOK REVIEWS.

Tweedie. By ISLA MAY MULLINS. Boston: The Page Co. Pp. 291. Price $1.50 net.

Those who have read the Blossom Shop Stories will welcome a new story by the same author. One must read the book to appreciate the charming personality of the young Swiss girl who goes to work in a professor's home.

Birds of Field, Forest and Park. By ALBERT FIELD GILMORE. Boston: The Page Co. Pp. 318. Price $2.50 net.

Although not a treatise on the science of ornithology, this book reproduces the atmosphere of the natural home of various birds most common in eastern North America by describing the conditions under which they are found and their usual surroundings as well as their looks, etc. There are about 150 varieties described, with 40 illustrations, a number of which are in full color.

First Lessons in Business. By J. A. BEXELL. Lippincott's Thrift Text Series. Philadelphia: J. B. Lippincott Co. Pp. 174. Price 68 cents.

This book presents in very interesting form a discussion of the business qualifications important to success. They include, among others, honesty, promptness and ambition. It then takes up the practical side of keeping personal and household accounts, the use of business forms, etc.

The book is intended for the eighth and ninth school years, and seems very well adapted for pupils of that age, as well as so sensible as to be interesting to older pupils.

First Course in Algebra. By WALTER BURTON FORD and CHARLES AMMERMAN. New York: The Macmillan Company. Pp. xiii + 334. Price $1.20.

While this text follows much the traditional order of topics, it introduces the modern spirit in many ways. The combination of the simple cases of factoring with multiplication and division is excellent, as is the free use of pictorial illustration, and the flexibility in exercises. This flexibility is brought about by starring in each list some that may be left for review or second year work, and referring to additional ones supplied in the appendix. The use of tables is also a good feature.

The text is good, the exercises well chosen, and the book as a whole seems teachable.

It is the first year book of a two-book series.

Projective Geometry. By OSWALD VEBLEIN and JOHN WESLEY YOUNG. Vol. II by Oswald Veblein. Boston: Ginn and Company. Pp. 511. Price $5.00.

This is an advanced course fitted for upper class and graduate students. It treats the subject from the abstract point of view, as did the first volume.

This volume, however, is devoted to special branches, in contrast to the first volume which treated the general subject.

The first chapter discusses the assumptions, and lays the foundation for the treatment of particular spaces. The following chapters take up the theorems according to class of space, and group in that space to which they belong.

Examples of the subjects treated are: Foundations of the Geometry of Reals; Relation of this Geometry to Complex Geometry; Affine and Metric Groups and the corresponding Geometries; Analysis Situs of Projective Spaces.

Junior High School Mathematics, First Book. By E. H. TAYLOR and FISKE ALLEN. New York: Henry Holt and Company. Pp. ix + 210.

This is the seventh grade book of a series for the junior high school. It follows the lines generally accepted as meeting the needs of this year, extending arithmetic and introducing the use of algebra formulas and the beginnings of geometry. Percentage and its application are given considerable attention.

Introductory Mathematical Analysis. By W. PAUL WEBBER and LOUIS CLARK PLANT. New York: John Wiley and Sons. Pp. xiii + 304.

In many ways this book seems to differ from the others that have combined the freshman mathematics into a single course.

It starts by bringing together the information and methods already acquired, reviewing elementary algebra briefly, and listing the geometric principles to be used. The next step is a discussion of some of the fundamentals underlying the course to be taken; this includes "Methods of Calculation," "Graphic Representation" and "Ratio, Proportion and Variation."

Coördinate systems follow, and algebra, trigonometry, analytical geometry, vector analysis and calculus are used in what seems like a logical and unusually inclusive course.

NOTES AND NEWS.

OTIS W. CALDWELL and S. A. Courtis have invited superintendents, supervisors and teachers, the country over, to coöperate with them in determining how well American school children of the present day can answer certain examination questions which originally were given in the Boston schools in 1845. The report made by the school committee at that time was sufficiently detailed to warrant the belief that interesting comparisons are possible in history, geography, arithmetic, grammar, vocabulary and natural philosophy.

Not all of the questions are valid for school work of the present day. However, a selection of 30 questions suitable for eighth grade children has been made, and test papers, instructions, etc., were being printed in the number of 50,000. Thirty thousand of these were to be distributed free among representative cities of the country, and the remainder to be sold at cost* to those who would return copies of their results for the general tabulations.† The tests were to be given between May 15 and June 1, and required about 100 minutes of class time. The scoring and tabulations are simple and take from four to eight hours, depending on the size of the class. Although it is too late to give these tests now, those interested may write to S. A. Courtis, 82 Eliot Street, Detroit, Mich., for further information concerning them. The coöperation of superintendents of small towns, country superintendents, and teachers of rural schools was particularly asked for.

THE spring meeting of the Philadelphia Section of the Association of Teachers of Mathematics of the Middle States and Maryland was held Saturday morning, April 5, in the William Penn High School.

* The cost of a full set of supplies including transportation both ways was about 65 cents for a class of 25 children.

† As soon as returns are received, a copy of the Boston standards will be sent. Later copies of the general tabulations will be provided. No results will be published by name without permission.

A brief business meeting was held, and the following officers for the coming year were elected. *President,* Professor John A. Miller, Swarthmore College; *Vice-President,* Dr. Harry M. Shoemaker, Northeast High School; *Secretary,* Grace L. Bentley, William Penn High School; *Executive Committee,* Ellen Pottberg, George L. McCracken.

Emily Renshaw, of the Department of Mathematics, Holmes Junior High School, gave a very interesting talk on "The Course in Mathematics of the Junior High School." Discussion followed.

Owing to the illness of Dr. Howard H. Mitchell, University of Pennsylvania, who was to have spoken on "The Use of Mathematics in the Preparation of Artillery Range Tables," Dr. George Gailey Chambers, University of Pennsylvania, gave a brief but very instructive talk on "The Application of Mathematics to Some Problems Related to College Entrance Requirements."

THE Southern Section met at The Johns Hopkins University on May 17.

The morning session was given to a paper on "Experimental Tests of Mathematical Ability," by Dr. Agnes L. Rogers, of Goucher College.

In the afternoon Dr. John H. Minnick, of The University of Pennsylvania, spoke on "Standardized Tests in Geometry," and Dr. Knight Dunlap, of The Johns Hopkins University, described the results obtained by using the army intelligence tests on a group of Johns Hopkins students.

The following officers were elected: *Chairman,* Dr. Clara Bacon, Goucher College; *Secretary-Treasurer,* Ida Hammond, McKinley Manual Training School, Washington; *Member of the Council,* Eugene R. Smith.